Longevity risk: past, present and future

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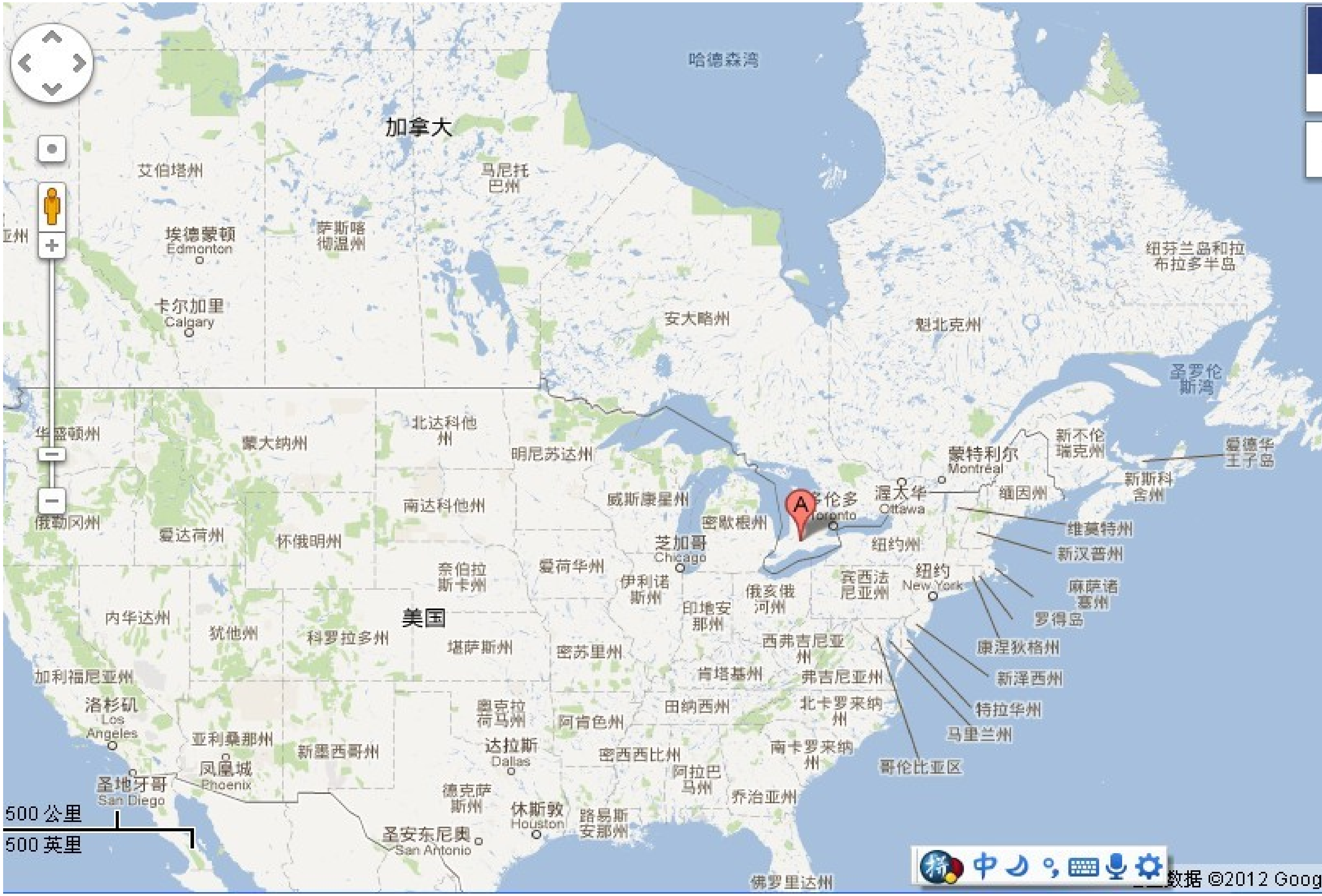

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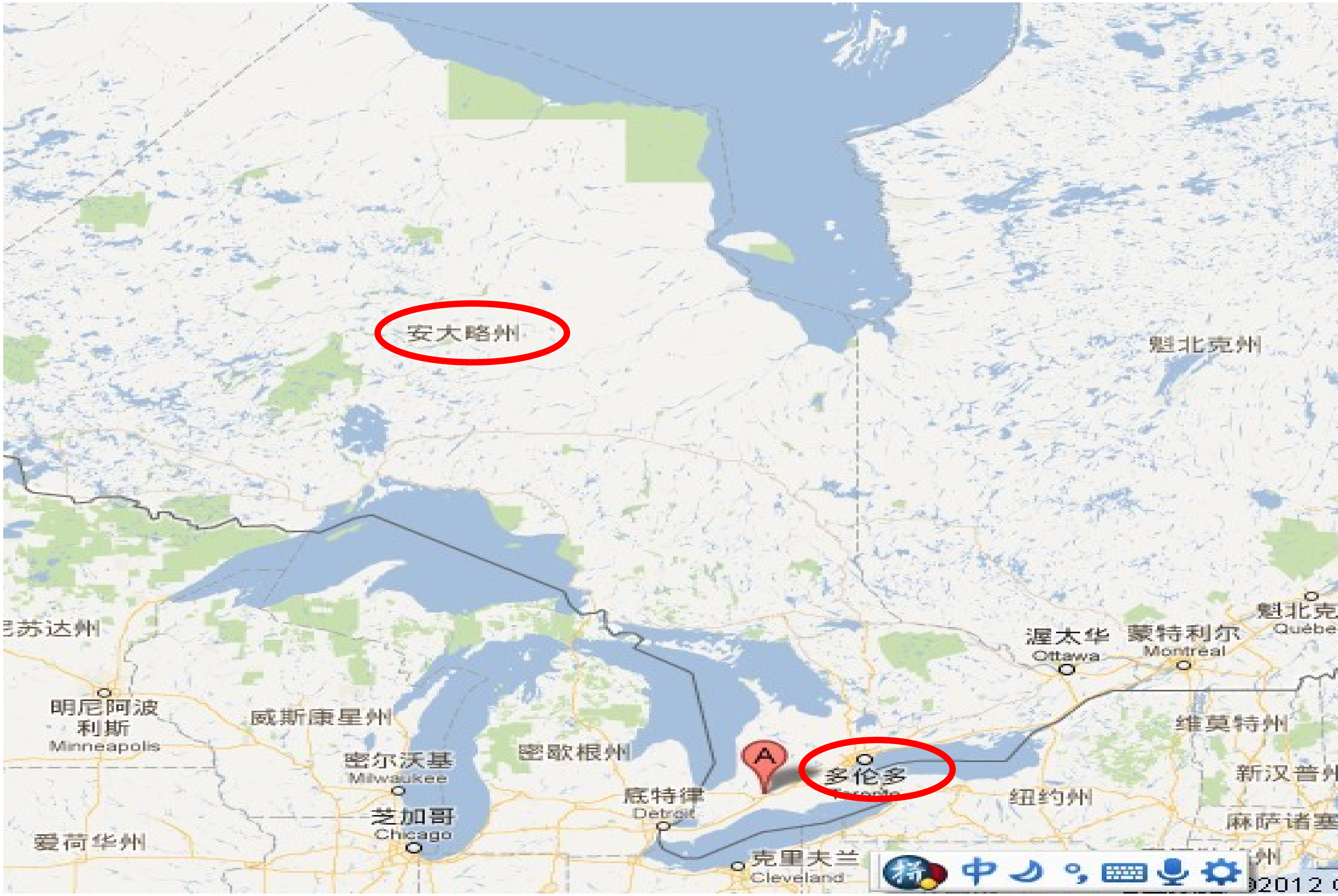


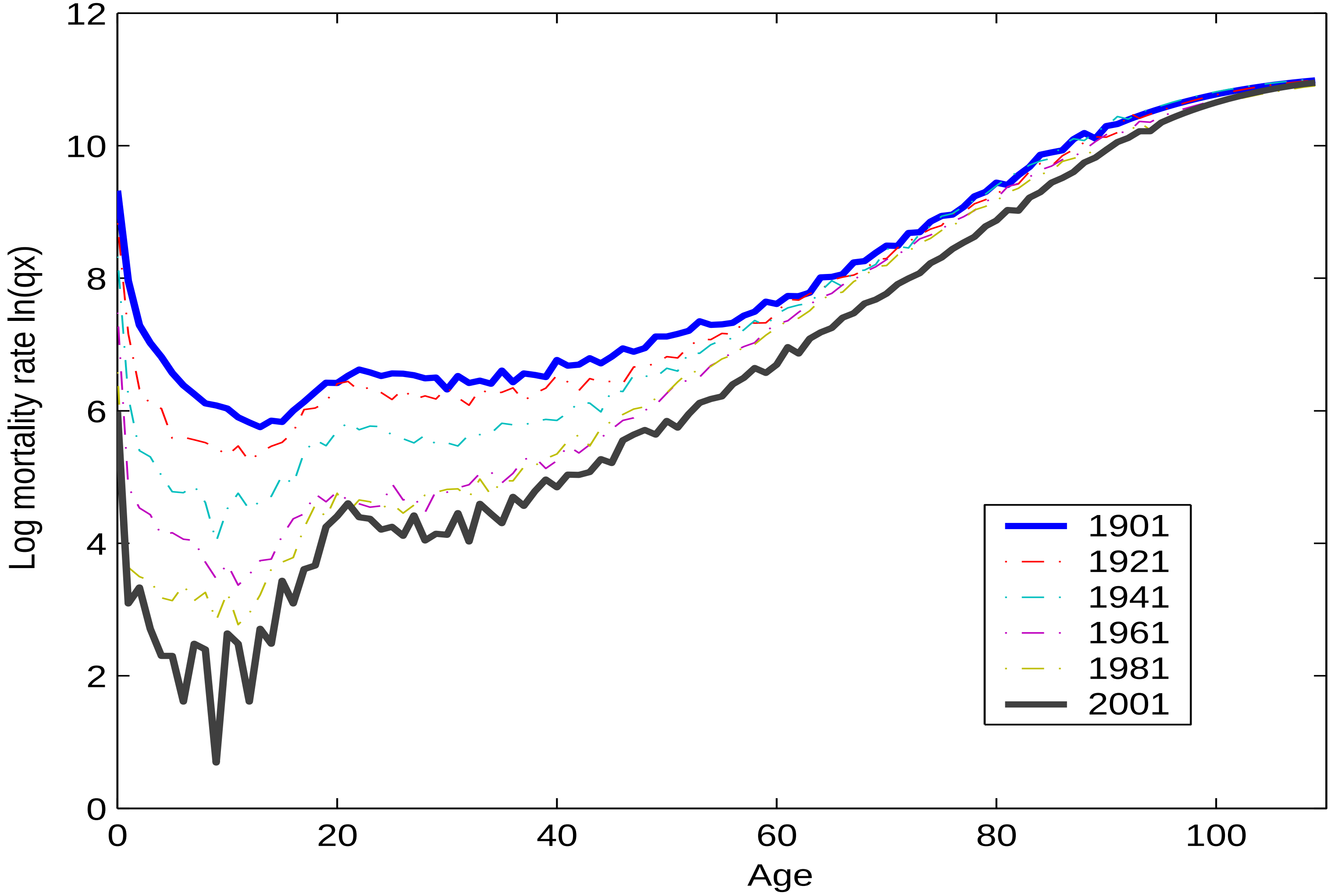
Outline

Past: The Meaning of Longevity Risk

Present: Stochastic Mortality Modelling
- Time Series Models
- Affine-Type Diffusion Models
- Subordinated Markov Mortality Model

Future: Risk Management of Longevity Risk

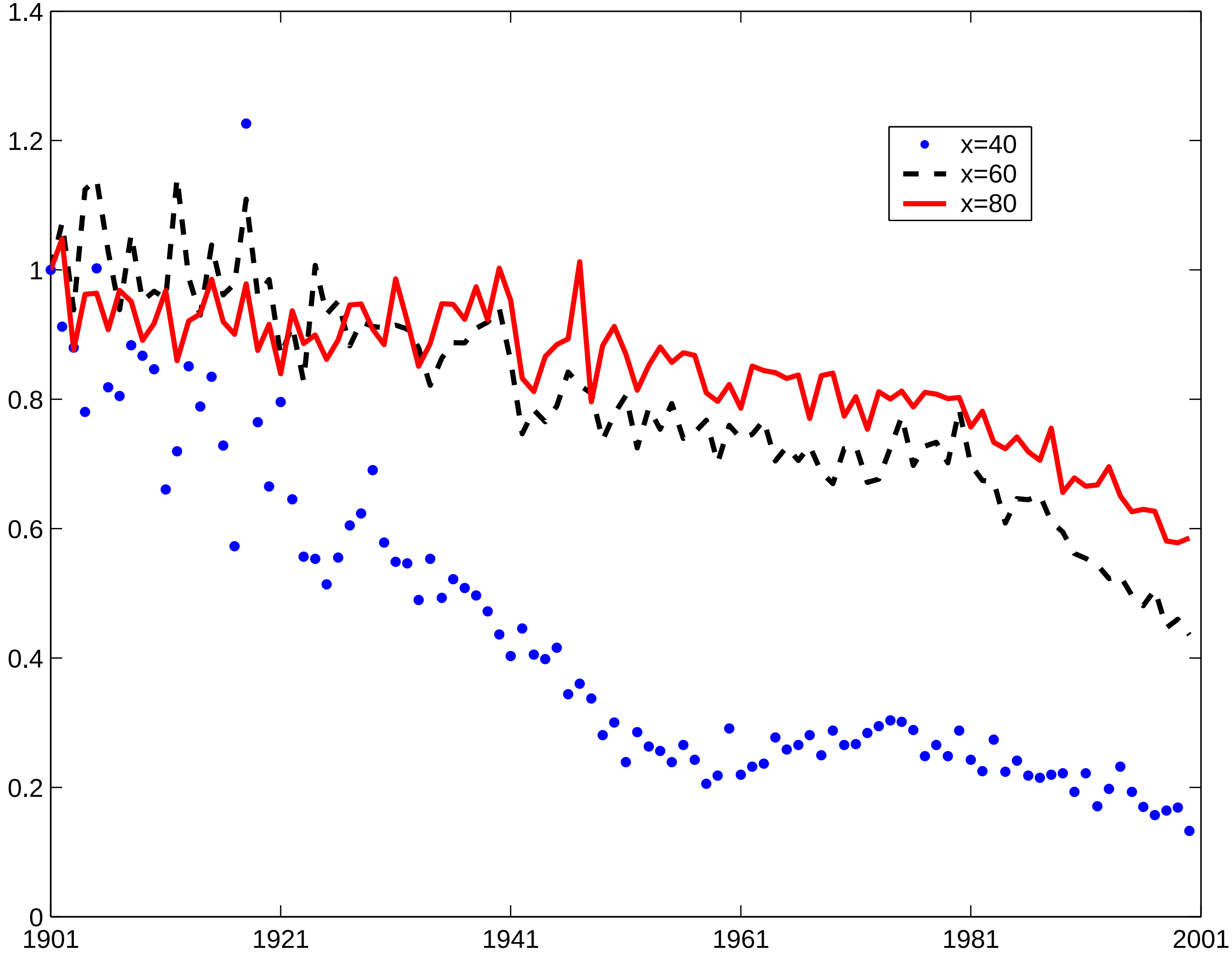
Historical Sweden Male Mortality Data



Mortality Changes at Different Ages

Ratio of

$\frac{q\_x(t)}{q\_x(1901)}$



Actuarial Perspective on Mortality Study

- Actuaries play a special role in ensuring adequate funds available to make the payments for insured products and social security systems.
- Actuarial calculation normally requires very long-term mortality prediction, say 60 years and above.
- Prospects of longer life are viewed as a positive change for individuals and as a substantial social achievement but unforeseen development could lead to unexpected spending and result in insolvency.
- The main challenge is not the longevity but instead the uncertainty contained in future mortality change.
- Stochastic mortality model is needed for prediction and pricing of mortality/longevity related products.

Model Selection Criteria

- Tractability: simple, transparent, easy to understand, tractable, etc.
- Meaningfulness: positive, biologically meaningful, consistent with historical data, etc.
- Applicability:
  - Parameter estimates should be robust relative to the period of data and range of ages employed.
  - Forecast levels of uncertainty and central trajectories should be plausible and consistent with historical trends and variability in mortality data.
  - It should be possible to use the model to generate sample paths and calculate prediction intervals.
  - At least for some countries, the model should incorporate a stochastic cohort effect.
  - The model should have a non-trivial correlation structure over age and time.

Stochastic Mortality Models

Different type of stochastic mortality models have been proposed since 1992.

- Time series models
  - The Lee-Carter model (Lee and Carter, 1992)
  - The Cairns-Blake-Dowd model (Cairns, Blake and Dowd, 2006)
- Affine-type diffusion models
  - Dahl, M. (2004), Biffis (2005), etc
  - Luciano and Vigna (2006), Non-mean reverting process
  - More: Milevsky and Promislow (2001), Ballotta and Haberman (2006), Liu, Mamon and Gao (2011)
- Markovian mortality models
  - The phase-type mortality model (Lin and Liu, 2007)
  - The subordinated Markov mortality model (Liu and Lin, 2012)

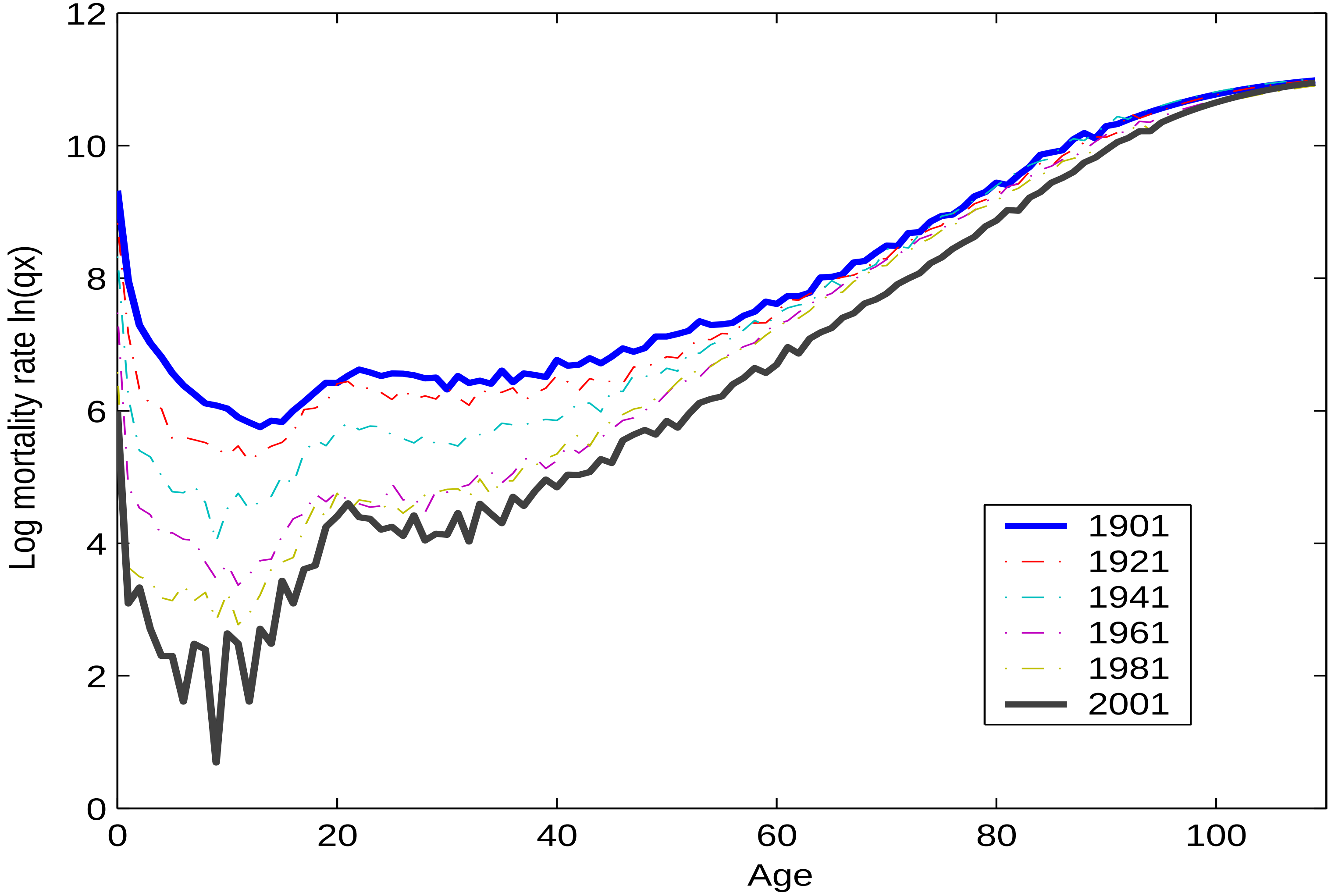
The Lee-Carter (LC) Model

- Let m_{xt} be the central mortality rate at age x in year t.
- The Lee-Carter model (1992)

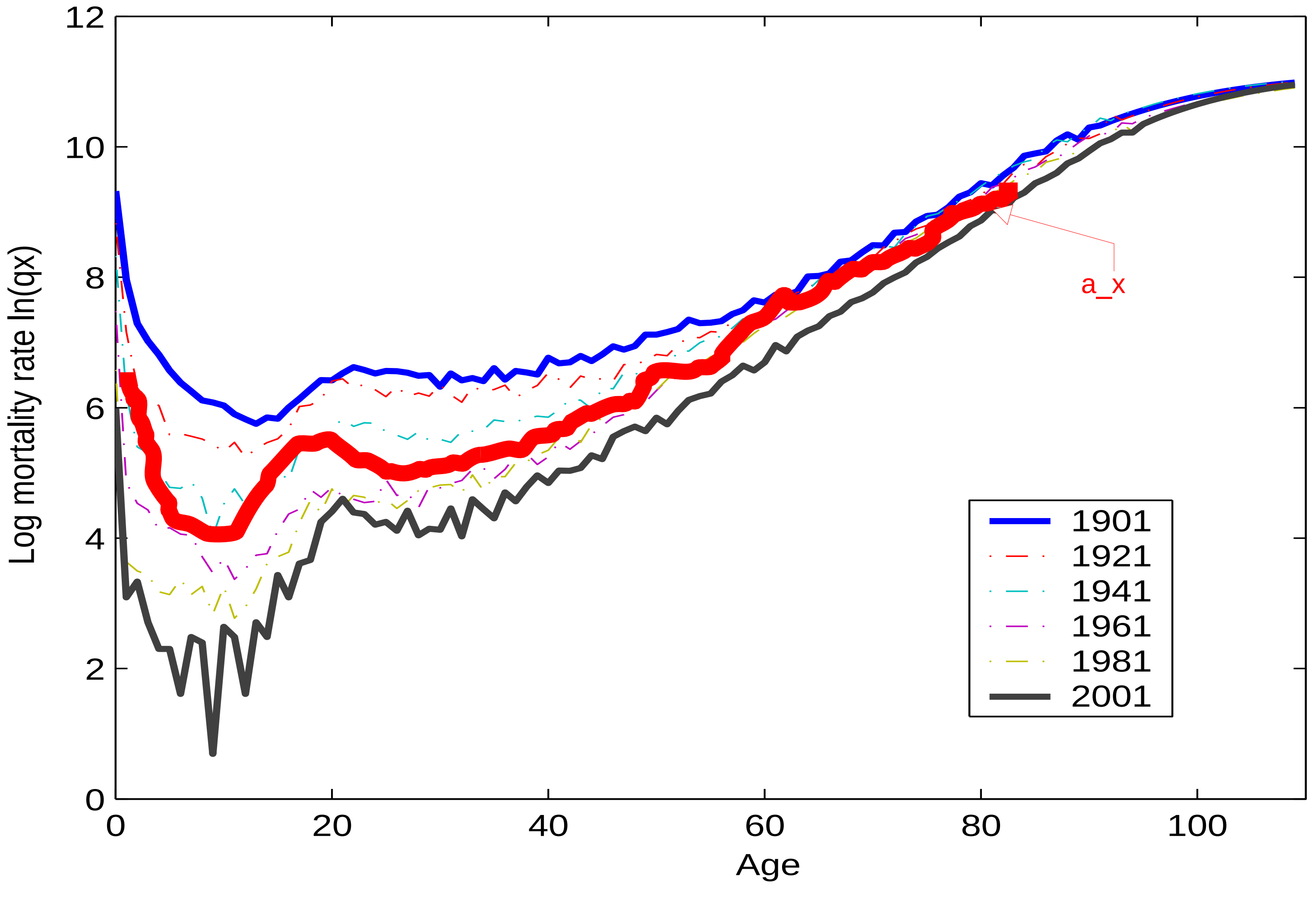
$$\log m_{xt} = a_x + b_x k_t + \epsilon_{xt} ,$$
$$k_t = k_{t-1} + c + \xi_t, \quad \text{with i.i.d } \xi_t \sim N(0, \sigma^2).$$

- Parameters interpretation
  - a_x is the general age shape of the $\log(m_{xt})$
  - b_x indicates the age response to the impact of k_t
  - k_t is a hidden stochastic process capturing the fluctuations in mortality random change
- Parameter estimation methods:
  - SVD method applied to $\log(m_{xt})$, Lee and Carter (1992)
  - MLE method applied to (D_{xt}, ETR_{xt}), Brouhns, Denuit and Vermunt (2002)

Applying the LC Model to Sweden Male Mortality Data



Applying the LC Model to Sweden Male Mortality Data



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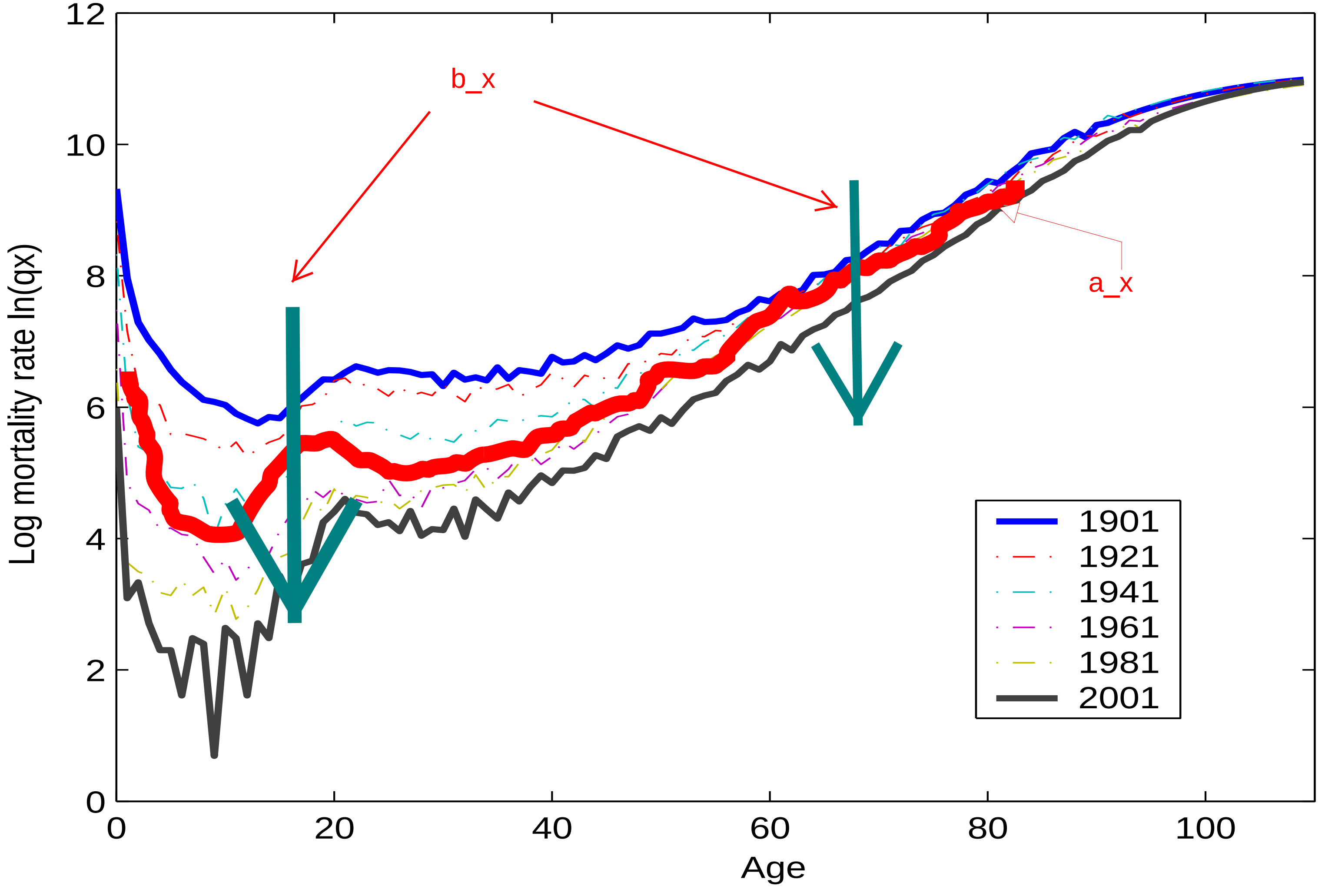
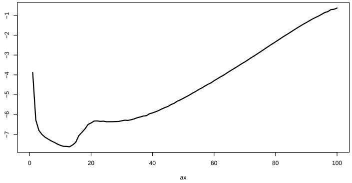
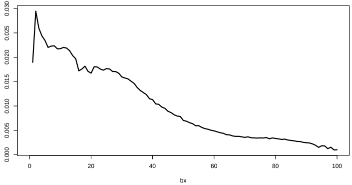


The Fitted Lee-Carter Parameters Using MLE method

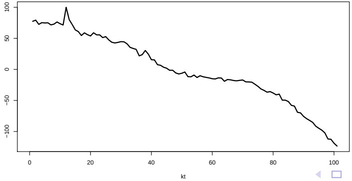
a_x



b_x



k_t



LC Prediction

- Now we have obtained a_x, b_x and the model for k_t using the data from year 0 to t_0.
- a_x and b_x will be treated as constants.
- The value of k_t at time $t_0 + n$, given the data available up to t_0, is predicted as follows:

$$\hat{k}_{t_0+n} = k_{t_0} + n \cdot \hat{c} + \sum_{j=1}^{n} \xi_j \, .$$

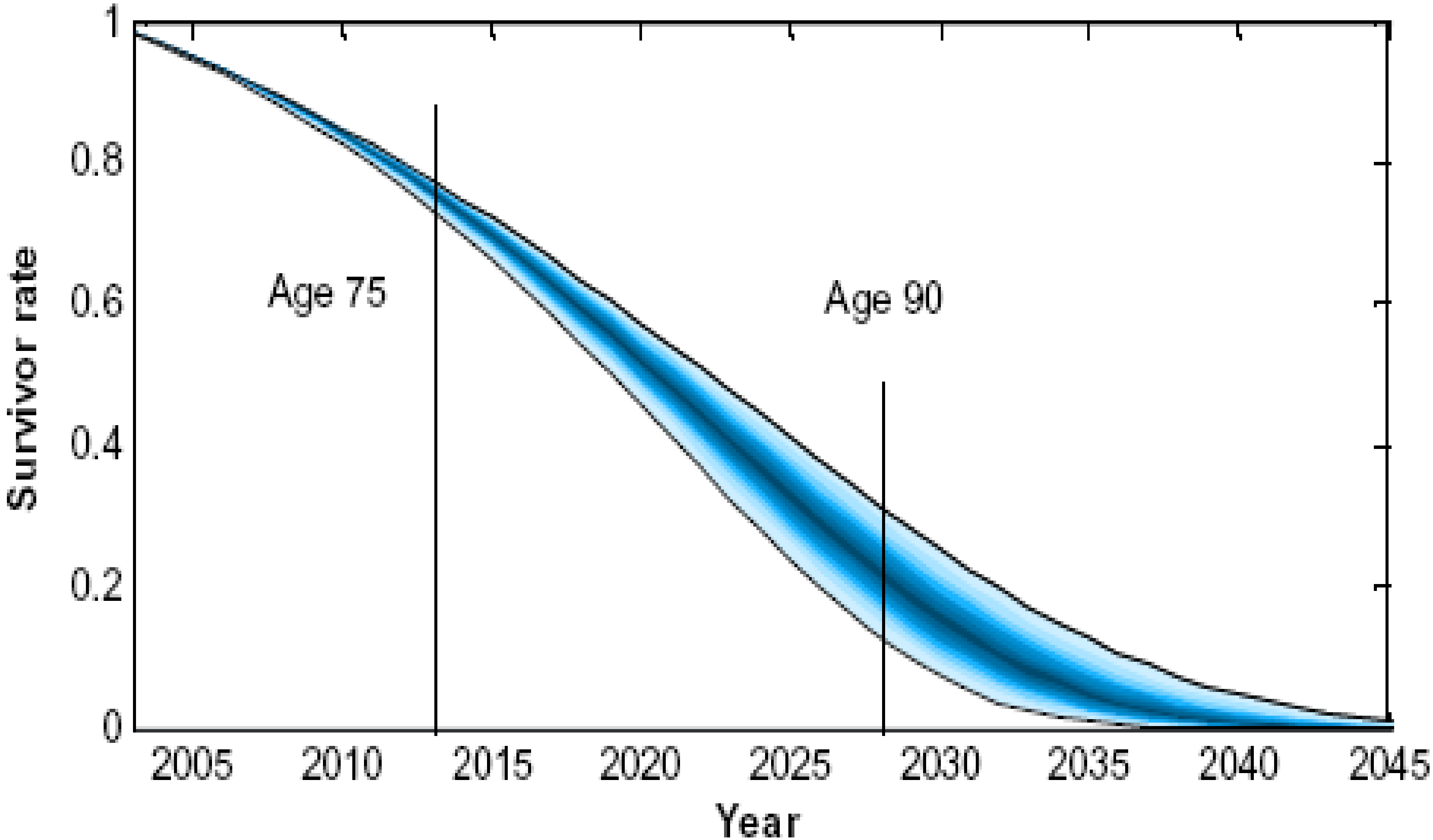
- The future mortality rates and other variables, such as the life expectancy at birth $e_0(t)$, can all be calculated.

$$m_{xt} = \exp(a_x + b_x \hat{k}_t)$$

$${}_n p_x = \prod_{j=0}^{n-1} p_{x+j} = \prod_{j=0}^{n-1} \exp(-m_{x+j,j})$$

$$S(x, s, t) = \frac{{}_t p_x}{{}_s p_x} \quad \text{for } s < t.$$

Survivor fan chart for 65-year old males in 2003 from cbdmodel.com



Advantages and Disadvantages of the LC model

- Simple, transparent, easy to use
- Fit to the historical data well
- Can be used for pricing and reserving calculation
- No explicit formula, simulation needed.
- Objective extrapolation, no need for expert opinion.

Affine-Type Diffusion Models—One Example

Our model is built on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q)$, where Q is a risk-neutral measure and $\mathcal{F}_t$ is the joint filtration generated by r_t and μ_t.

- The short rate process r_t follows a Vasicek model

$$dr_t = a(b - r_t)dt + \sigma dW_t^1,$$

  where a, b and σ are positive constants and W_t^1 is a standard Brownian motion.

- The force of mortality μ_t follows a non-mean reverting process, justified in Luciano and Vigna (2006)

$$d\mu_t = c\mu_t dt + \xi dZ_t,$$

  where c and ξ are positive constants and Z_t is a standard Brownian motion correlated with W_t^1 so that

$$dW_t^1 dZ_t = \rho dt.$$

  In other words, $Z_t = \rho W_t^1 + \sqrt{1-\rho^2} W_t^2$, where W_t^2 is a standard Brownian motion independent of W_t^1.

No-Arbitrage Evaluation Approach

- We adopt the No-Arbitrage approach for the evaluation of life annuity contract and GAOs.
- For a life aged x at time 0, under the Q measure:

$$
\begin{aligned}
M(T, T+n) &= \mathbf{E}^Q\left[e^{-\int_T^{T+n} r_u du} \cdot \mathbf{I}_{\{\tau \geq T+n\}} \middle| \mathcal{F}_T \right] \\
&= \mathbf{I}_{\{\tau \geq T\}} \cdot \mathbf{E}^Q\left[e^{-\int_T^{T+n} r_u du} e^{-\int_T^{T+n} \mu_v dv} \middle| \mathcal{F}_T \right], \\
a_x(T) &= \mathbf{I}_{\{\tau \geq T\}} \sum_{n=0}^{\infty} \mathbf{E}^Q\left[e^{-\int_T^{T+n} r_u du} e^{-\int_T^{T+n} \mu_v dv} \middle| \mathcal{F}_T \right], \\
c(t, T) &= \mathbf{E}^Q\left[e^{-\int_t^{T} r_u du} \mathbf{I}_{\{\tau \geq T\}} (a_x(T) - K)^+ \middle| \mathcal{F}_t \right] \\
&= \mathbf{I}_{\{\tau \geq t\}} \mathbf{E}^Q\left[e^{-\int_t^{T} r_u du} e^{-\int_t^{T} \mu_v dv} (a_x(T) - K)^+ \middle| \mathcal{F}_t \right].
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\end{aligned}
$$

S(x,T,T+n)

Liu, Mamon and Gao (2011)

"A comonotonicity-based valuation method for annuity-linked contracts"

- Use the change of numéraire technique twice to simplify the expression.

$$M(T, T+n) = \mathbf{I}_{\{\tau \geq T\}} \cdot \mathbf{E}^{Q}\left[e^{-\int_T^{T+n} r_u du} e^{-\int_T^{T+n} \mu_v dv} \middle| \mathcal{F}_T \right],$$

$$M(T, T+n) = \mathbf{I}_{\{\tau \geq T\}} B(T, T+n) \mathbf{E}^{\widetilde{Q}}\left[e^{-\int_T^{T+n} \mu(v) dv} \middle| \mathcal{F}_t \right]$$

$$a_x(T) = \sum_{n=0}^{\infty} \beta(T, T+n) e^{-\left(A(T,T+n) r_T + \widetilde{G}(T,T+n)\mu_T\right)}$$

$$c(t, T) = \mathbf{I}_{\{\tau \geq t\}} \mathbf{E}^{Q}\left[e^{-\int_t^{T} r_u du} e^{-\int_t^{T} \mu_v dv} (a_x(T) - K)^+ \middle| \mathcal{F}_t \right]$$

$$= M(t, T) \underbrace{\mathbf{E}^{\widehat{Q}}\left[(a_x(T) - K)^+ \middle| \mathcal{F}_t \right]}_{\text{To derive its comonotonic bounds}}$$

Advantages and Disadvantages of Affine-type models

- Mathematical tractability
- Well-developed methodology available to be used
- Lack of biological or empirical data evidence to support the use of this type of models.

Subordinated Markov Mortality Model

- Lin, X. S. and Liu, X. (2007), Markov aging process and phase-type law of mortality, *North American Actuarial Journal* 11, 92 – 109.
- Markov Aging Process and Phase-type Mortality Model
  - reflects the historic mortality experience;
  - is tractable mathematically, utilizing matrix analytic techniques.
  - has biological interpretation.

Subordinated Markov Mortality Model (cont.)

- Use a subordinating stochastic process (time-change) to incorporate stochastic mortality such that the stochastic model
  - has desirable properties: longevity risk is reflected in the model and confidence bands of future mortality rates are of banana-shape;
  - remains mathematically tractable.

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- Add risk loading parameters to the model for the pricing of mortality linked securities so that
  - we can calibrate the model to market information;
  - the price of basic mortality-linked securities (caplets and floorlets) has a closed form.

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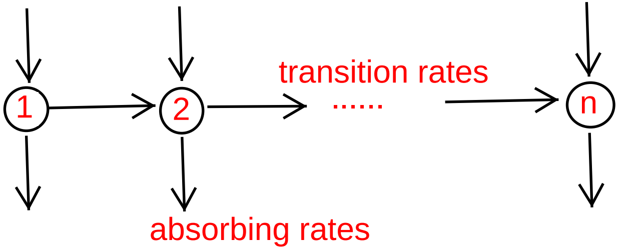
- Liu, X. and Lin, X.S. (2012), A Subordinated Markov Model for Stochastic Mortality, *European Actuarial Journal* 2(1): 105-127

The Baseline Model

- Assume the aging process of life (x) follows a finite-state continuous-time Markov process $\{J_t;\ t \geq 0\}$.
- The state space of the Markov process is assumed to consist of a set of transient states $E = \{1, 2, \cdots, n\}$ that represent chronological health statuses before death and a single absorbing state Δ representing the death.



The Baseline Model

- The intensity matrix for the transient states is thus given by

$$\mathbf{\Lambda} = \begin{pmatrix} -\lambda_1 & {\lambda_1}' & 0 & \cdots & 0 \\ 0 & -\lambda_2 & {\lambda_2}' & \cdots & 0 \\ 0 & 0 & -\lambda_3 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda_n \end{pmatrix},$$

  where

$$\lambda_i = {\lambda_i}' + q_i.$$

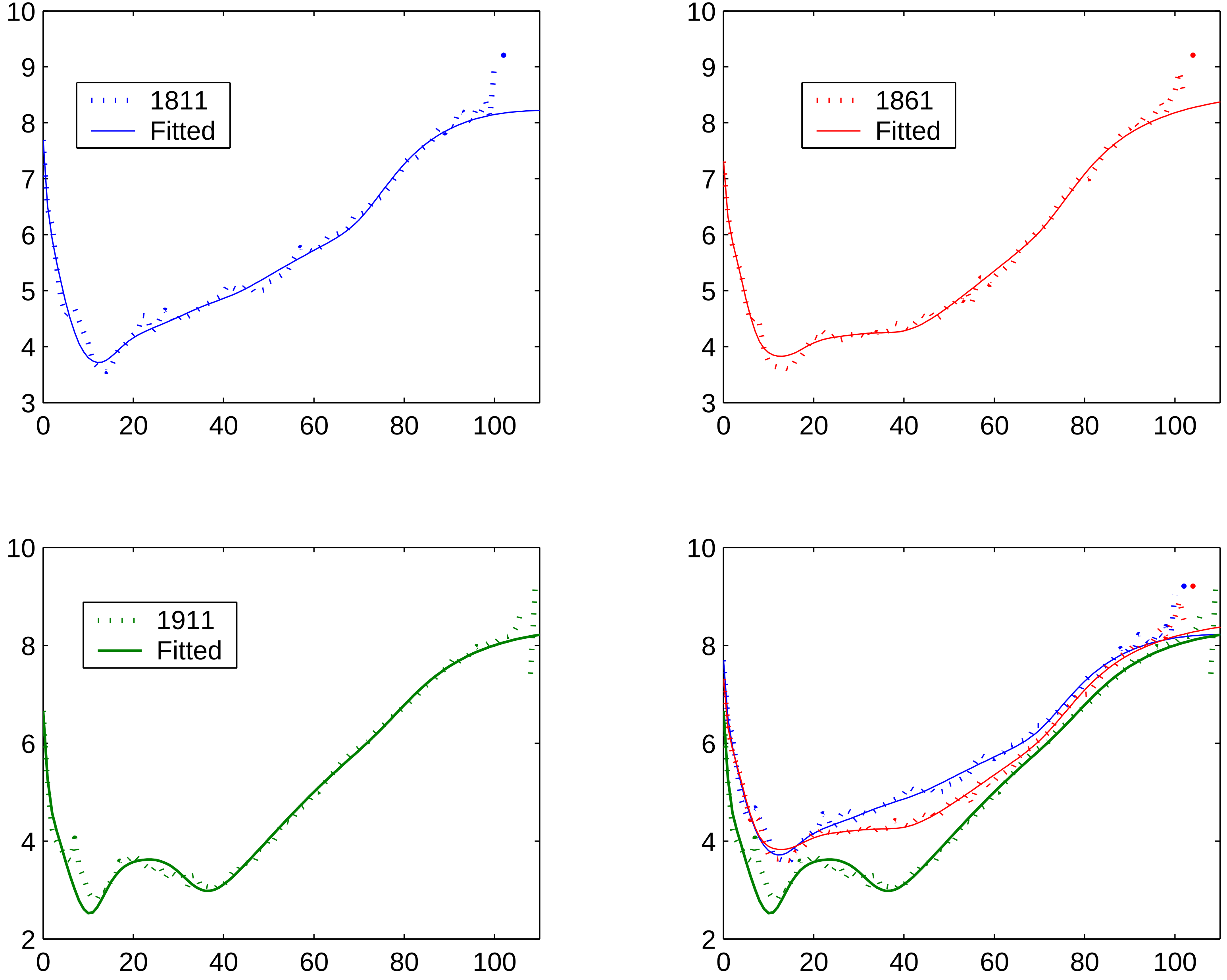
- ${\lambda_i}' > 0$ denotes the aging rate from status i to status $i+1$, $q_i > 0$ denotes the death rate of the life given that the life is at status i.

Estimated aging related parameters

Table 1: Estimated aging related parameters for Swedish cohorts of year 1811, 1861, and 1911

Year	Parameters					
	λ	b	a	$[i_1, i_2]$	q	p
1811	2.5657	3.1504e-03	1.9888e-03	[42, 99]	9.3157e-09	3
1861	2.4794	4.4825e-03	1.9033e-03	[42, 89]	2.6351e-13	5
1911	2.3707	9.0987e-04	2.8939e-03	[33, 70]	1.8872e-15	6

Fitted curves on Sweden cohort 1811 to 1911



The Baseline Model

- Let $T(x)$ denote the time till absorption (death) of the Markov process. $T(x)$ has a phase-type distribution with phase-type representation $(\boldsymbol{\alpha}, \boldsymbol{\Lambda})$ of order n.
- The survival function of $T(x)$ is

$$S_0(t) = \boldsymbol{\alpha}\, e^{\boldsymbol{\Lambda} t}\mathbf{e}, \quad t > 0.$$

- The survival function of $T(x+s)$ is

$$\frac{S_0(t+s)}{S_0(s)} \;=\; \boldsymbol{\alpha}_s e^{\boldsymbol{\Lambda} t}\mathbf{e}, t > 0,$$

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- Note: Survival distribution is a deterministic function.

Gamma Process and Subordinated Aging Process

- The gamma subordinating process:
  - $\gamma_0 = 0$;
  - it has independent increments, i.e., for any partition $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables $\gamma_{t_1} - \gamma_{t_0}, \cdots, \gamma_{t_n} - \gamma_{t_{n-1}}$ are mutually independent; and
  - the increment $\gamma_{t+s} - \gamma_t$ has a gamma distribution with mean s and variance νs, for any $s, t \geq 0$.
- The aging process J_t is subordinated by the gamma process and the resulting aging process Z_t is now a subordinated Markov process
$$Z_t = J_{\gamma_t}.$$

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- Interpretation: Allow aging process be altered by external factors randomly

Survival Index

- With the model, we have

$$S(t) = S_0(\gamma_t) = \boldsymbol{\alpha}\, e^{\boldsymbol{\Lambda}\gamma_t}\, \mathbf{e}, \quad t > 0.$$

- The new survival function $S(t)$ is a stochastic process and is referred to as the survival index for the cohort under consideration.
- Thus the new mortality model is a stochastic mortality model.

Term Structure of Mortality

- For $0 \leq s \leq t$, let $P(s,t)$ be the survival function of a life aged x at time 0 to be alive from time s to time t that is measured at time s. $\{P(s,t); 0 \leq s \leq t\}$ is commonly referred to as the term structure of stochastic mortality.
- $$P(s,t) = \frac{1}{S(s)} E\left[S(t) \middle| \mathcal{F}_s\right],$$

  where $\mathcal{F}_t, t \geq 0$, is the filtration generated by $S(t)$.
- We have shown

  $$P(s,t) = \boldsymbol{\alpha}_{\gamma_s}\, e^{\tilde{\boldsymbol{\Lambda}}(t-s)}\mathbf{e},\ 0 \leq s \leq t.$$

  As a special case, the term structure at time 0 is given by

  $$P(0,t) = \boldsymbol{\alpha}\, e^{\tilde{\boldsymbol{\Lambda}}t}\mathbf{e},\ t \geq 0.$$

Explicit Expression of Term Structure of Mortality

Suppose that the eigenvalues $-\lambda_1, \cdots, -\lambda_n$ of the intensity matrix $\mathbf{\Lambda}$ are distinct. Let $\mathbf{h_1}, \cdots, \mathbf{h_n}$ and $\boldsymbol{\nu}_\mathbf{1}, \cdots, \boldsymbol{\nu}_\mathbf{n}$ be their corresponding right and left eigenvectors such that $\boldsymbol{\nu}_\mathbf{i}\mathbf{h_i} = 1$. It is known that $\boldsymbol{\nu}_\mathbf{i}\mathbf{h_j} = 0,\ i \neq j,\ i,j = 1,\ldots,n$. Then, $P(s,t)$ has the phase-type representation $(\boldsymbol{\alpha}_{\gamma_s}, \tilde{\mathbf{\Lambda}})$, where

$$\boldsymbol{\alpha}_{\gamma_s} = \frac{\boldsymbol{\alpha}\, e^{\mathbf{\Lambda}\gamma_s}}{\boldsymbol{\alpha} e^{\mathbf{\Lambda}\gamma_s}\mathbf{e}},$$

and

$$\tilde{\mathbf{\Lambda}} = -\sum_{i=1}^{n} \tilde{\lambda}_i\, \mathbf{h_i}\, \boldsymbol{\nu}_\mathbf{i},$$

with $\tilde{\lambda}_i$ being given by

$$\tilde{\lambda}_i = \frac{1}{\nu} \ln(1 + \nu\lambda_i).$$

Variance of Survival Index

The variance of $S(t)$ is given by

$$Var\left[S(t)\right] = (\boldsymbol{\alpha} \otimes \boldsymbol{\alpha}) \left[e^{\left(\widetilde{\boldsymbol{\Lambda} \oplus \boldsymbol{\Lambda}}\right) t} - e^{\left(\tilde{\boldsymbol{\Lambda}} \oplus \tilde{\boldsymbol{\Lambda}}\right) t} \right] (\mathbf{e} \otimes \mathbf{e}) \, .$$

Matrix analytic methodology

- Denote $\mathbf{D} = diag(-\lambda_1, \cdots, -\lambda_n)$, then

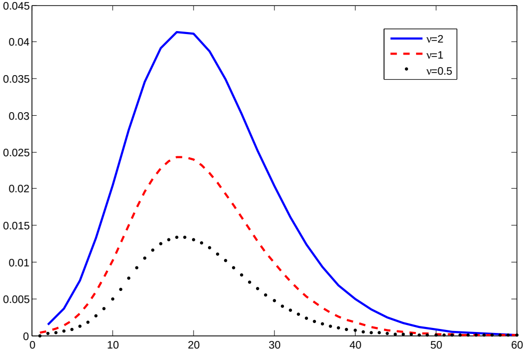
$$\mathbf{D} \oplus \mathbf{D} = diag(\mathbf{D} - \lambda_1 \mathbf{I}, \mathbf{D} - \lambda_2 \mathbf{I}, \cdots, \mathbf{D} - \lambda_n \mathbf{I}),$$

  the Kronecker sum of $\mathbf{D}$ to itself, is diagonal with diagonal entries $-\zeta_k, \; k = 1, \cdots, n^2$, where $\zeta_{i+j} = \lambda_i + \lambda_j, \; i, j = 1, \cdots, n$.
- Denote $\widetilde{\mathbf{D} \oplus \mathbf{D}} = diag(-\tilde{\zeta}_1, \cdots, -\tilde{\zeta}_{n^2})$ and

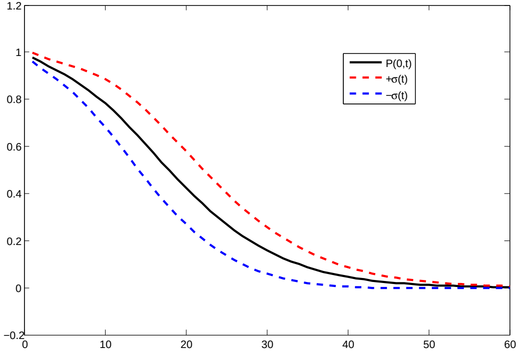
$$\widetilde{\mathbf{\Lambda} \oplus \mathbf{\Lambda}} = (\mathbf{H} \otimes \mathbf{H}) \left(\widetilde{\mathbf{D} \oplus \mathbf{D}} \right) (\mathbf{H} \otimes \mathbf{H})^{-1},$$

  where $\mathbf{H} = (\mathbf{h_1}, \cdots, \mathbf{h_n})$ and $\otimes$ is the symbol for the Kronecker product.

Variance function $Var[S(t)]$, $t \geq 0$, for $\nu = 0.5$, 1 and 2



Term structure $P(0, t)$ with one-σ confidence intervals, based on $\nu = 1$



Interpretation of Parameter ν

- The curve of the term structure $P(0, t), \; t \geq 0$, exhibits a twisted upward shift as the value of ν increases.
- The variance function $Var[S(t)], \; t \geq 0$, increases as ν gets larger.
- As a result, parameter ν may be interpreted as the level of longevity risk or the longevity parameter.

Development of Stochastic Mortality Modelling

From 4 talks in IME2006 to 4 sessions in IME2012.

Future: Risk Management of Longevity Risk

Could be my next year's topic at CICIRM.