Optimal Reciprocal Reinsurance Treaties between Insurers and Reinsurers

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1 Reinsurance treaties

Let X be the (aggregate) loss or claim for an insurer in a fixed time period. We assume that X is a non-negative random variable with distribution function $F(x) = \Pr\{X \leq x\}$, survival function $S(x) = 1 - F(x) = \Pr\{X > x\}$, mean $\mathrm{E}X = \mu > 0$, and variance $\mathrm{Var}X = \sigma^2 > 0$.

Under a reinsurance treaty, a reinsurer will cover the part of the loss, say $f(X)$ with $0 \leq f(X) \leq X$, and the insurer will retain the rest of the loss, which is denoted by $I_f(X) = X - f(X)$, where the function $f(x)$, satisfying $0 \leq f(x) \leq x$, is known as a ceded loss function and the function $I_f(x) = x - f(x)$ is called a retained loss function. The losses $f(X)$ and $I_f(X)$ are called ceded loss and retained loss, respectively.

Under the reinsurance contract f, we denote the reinsurance premium by P_R^f. Thus, the net insurance premium received by the insurer is $P_I^f = P_0 - P_R^f$, where P_0 is the insurance premium received by the insurer from an insured. Therefore, under the reinsurance treaty f, the insurer and reinsurer share both the insurance loss $X = I_f(X) + f(X)$ and the insurance premium $P_0 = P_I^f + P_R^f$.

Mathematically, an optimal reinsurance is a specified form of the ceded loss $f(X)$, which satisfies a certain optimization criterion. Optimal forms of reinsurance depend on optimization criteria and reinsurance premium principles. Several optimization criteria commonly used in the study of optimal reinsurance are

- maximizing the expected utility function of a company's wealth;
- minimizing the variance of a company's risk;

- minimizing the risk measures of a company's risk;
- and minimizing the ruin probability or equivalently maximizing the survival probability of a company.

Most existing results on optimal reinsurance are from an insurer's point of view.

Arrow (1963) showed that when the optimization criterion is to maximize the expected concave utility function of an insurer's wealth with a given expected ceded loss, the optimal reinsurance for an insurer is a stop-loss reinsurance.

It is well-known that the optimal reinsurance for an insurer, which minimizes the variance of the insurer's loss with a given expected ceded loss, is also a stop-loss reinsurance.

However, Vajda (1962) showed that the optimal reinsurance for a reinsurer, which minimizes the variance of the reinsurer's loss with a fixed net reinsurance premium, is a quota-share reinsurance among a class of ceded loss functions that include stop-loss reinsurance contracts.

Kaluszka and Okolewski (2008) showed that if an insurer wants to maximize his expected utility with the maximal possible claim premium principle, the optimal form of reinsurance for the insurer is a limited stop-loss reinsurance.

Cai, et al. (2008) proved that depending on the risk measures level of confidence, the optimal reinsurance for an insurer, which minimizes the value-at-risk (VaR) and the conditional tail expectation (CTE) of the total risk of the insurer, can be in the forms of a stop-loss reinsurance or a quota-share reinsurance or the combination of a quota-share reinsurance and a stop-loss reinsurance under the expected value principle and among the increasing convex ceded loss functions.

It is interesting to notice that most optimal forms of reinsurance in these cited papers are stop-loss reinsurance contracts.

However, an optimal reinsurance contract for an insurer may not be optimal for a reinsurer and it might be unacceptable for a reinsurer as pointed out by Borch (1969). An interesting question about optimal reinsurance is to design a reinsurance so that it considers the interests of both an insurer and a reinsurer and it is fair, in some sense, to both of the parties. Borch (1960) first considered this issue. He discussed the optimal quota-share retention and stop-loss retention that maximize the product of the expected utility functions of the two parties' wealth.

2 Reciprocal reinsurance treaties

We consider the interests of both an insurer and a reinsurer under a reinsurance treaty f by studying their joint survival probability, which is denoted by

$$J_S^f = \Pr\{I_f(X) \le P_I^f + u_I,\ f(X) \le P_R^f + u_R\},$$

where $u_I > 0$ and $u_R > 0$ are the initial wealth of the insurer and reinsurer, respectively; and their joint profitable probability, which is defined by

$$J_P^f = \Pr\{I_f(X) \le P_I^f,\ f(X) \le P_R^f\}.$$

We point out that mathematically, the joint profitable probability can be viewed as the special case of the joint survival probability by setting $u_I = 0$ and $u_R = 0$. However, the two joint probabilities have different economic interpretations. When we use them as objective functions, solutions for optimal reinsurance treaties are different.

Under a reinsurance premium principle π, let $\mathcal{F}^{\pi}$ denote the class of admissible reinsurance policies, which consists of all ceded loss functions $f(x)$ satisfying $0 \le f(x) \le x$ for $x \ge 0$.

Furthermore, to avoid some tedious discussions, we assume that the loss random variable X has a continuous and strictly increasing distribution function on $(0, \infty)$ with a possible mass at 0.

In a quota-share reinsurance, $f(X) = (1-b)X$ and $I_f(X) = bX$, where $0 \le b \le 1$ is the quota-share retention. We denote the reinsurance premium, the net insurance premium, the joint survival probability, and the joint profitable probability for the quota-share reinsurance by $P_R(b) = P_R^f|_{f(X)=(1-b)X}$, $P_I(b) = P_I^f|_{f(X)=(1-b)X}$, $J_S(b) = J_S^f|_{f(X)=(1-b)X}$ and $J_P(b) = J_P^f|_{f(X)=(1-b)X}$, respectively.

Under a stop-loss reinsurance treaty, $f(X) = (X - d)_+ = \max\{X - d,\ 0\}$ and $I_f(X) = X \wedge d = \min\{X,\ d\}$, where $d \geq 0$ is the stop-loss retention. We denote the reinsurance premium, the net insurance premium, the joint survival probability, and the joint profitable probability for the stop-loss reinsurance by $P_R(d) = P_R^f|_{f(X)=(X-d)_+}$, $P_I(d) = P_I^f|_{f(X)=(X-d)_+}$, $J_S(d) = J_S^f|_{f(X)=(X-d)_+}$, and $J_P(d) = J_P^f|_{f(X)=(X-d)_+}$, respectively.

In a limited stop-loss reinsurance contract, $f(X) = (X - d_1)_+ \wedge d_2$ and $I_f(X) = X - (X - d_1)_+ \wedge d_2$, where $d_1 \geq 0$ is a threshold level and the reinsurer will cover the loss over the level d_1 while $d_2 \geq 0$ is the largest loss the reinsurer would like to cover. The retention in the limited stop-loss reinsurance is the vector (d_1, d_2) with $d_1 \geq 0$ and $d_2 \geq 0$. We denote the reinsurance premium, the net insurance premium, the joint survival probability, and the joint profitable probability for the limited stop-loss reinsurance by $P_R(d_1, d_2) = P_R^f|_{f(X)=(X-d_1)_+ \wedge d_2}$, $P_I(d_1, d_2) = P_I^f|_{f(X)=(X-d_1)_+ \wedge d_2}$, $J_S(d_1, d_2) = J_S^f|_{f(X)=(X-d_1)_+ \wedge d_2}$, and

$J_P(d_1, d_2) = J_P^f|_{f(X)=(X-d_1)_+\wedge d_2}$, respectively.

3 Optimal reinsurance retentions under the expected value principle

In this section, we assume that the reinsurance premium is determined by the expected value principle. Under this principle, the reinsurance premium $P_R^f = (1+\theta_R)\mathrm{E}f(X)$, where $\theta_R > 0$ is the relative safety loadings for the reinsurer. Thus, in the quota-share reinsurance,

$$P_R(b) = (1+\theta_R)(1-b)\mu \quad \text{and} \quad P_I(b) = P_0 - (1+\theta_R)(1-b)\mu.$$

In the stop-loss reinsurance,

$$\begin{aligned} P_R(d) &= (1+\theta_R)\mathrm{E}(X-d)_+ = (1+\theta_R)\int_d^\infty S(x)\mathrm{d}x, \\ P_I(d) &= P_0 - (1+\theta_R)\int_d^\infty S(x)\mathrm{d}x. \end{aligned}$$

We define $p = (1 + \theta_R)\mu - P_0$.

We first determine the optimal quota-share retention $b^* \in [0, 1]$ satisfying $J_S(b^*) = \max_{b\in[0,1]} J_S(b)$.

Theorem 1. *(1) If $u_I = p$, the optimal quota-share retention is $b^* = 0$ satisfying $J_S(0) = \max_{0\le b\le 1} J_S(b) = F(u_I + u_R + P_0)$.*

(2) If $u_I < p$, the optimal quota-share retention is $b^* = 1$ satisfying $J_S(1) = \max_{0\le b\le 1} J_S(b) = F(u_I + P_0)$.

(3) If $u_I > p$, the optimal quota-share retention is

$$b^* = b_0 = \frac{u_I - p}{u_I + u_R - p} \tag{1}$$

satisfying $J_S(b^*) = \max_{0\le b\le 1} J_S(b) = F\big(u_I + u_R + P_0\big)$. □

Remark 1. In the quota-share reinsurance, we denoted the survival probabilities of the insurer and reinsurer by $S_I(b) = \Pr\{bX \leq P_I(b) + u_I\}$ and $S_R(b) = \Pr\{(1-b)X \leq P_R(b) + u_R\}$. Thus, under the expected value principle, for $0 < b < 1$, $S_I(b) = F\big((1+\theta_R)\mu + \frac{u_I - p}{b}\big)$ and $S_R(b) = F\big((1+\theta_R)\mu + \frac{u_R}{1-b}\big)$.

It is interesting to notice that under the optimal quota-share retention b^* given by (1), the insurer and the reinsurer have the same survival probability with $S_I(b^*) = S_R(b^*) = F\big(u_I + u_R + P_0\big)$. $\square$

We then consider the joint profitable probability $J_P(b)$. In this case, we obtain the following result.

Proposition 1. *If* $p = 0$*, the optimal quota-share retention is* $b^* = b$ *for any* $b \in [0, 1]$ *satisfying* $J_P(b^*) = \max_{0 \le b \le 1} J_P(b) = F(P_0)$*. If* $p \neq 0$*, the optimal quota-share retention is* $b^* = 1$ *satisfying* $J_P(1) = \max_{0 \le b \le 1} J_P(b) = F(P_0)$. □

We then determine the optimal stop-loss retention $d^* \in [0, \infty)$ satisfying $J_S(d^*) = \max_{d \in [0, \infty)} J_S(d)$.

Theorem 2. *The optimal stop-loss retention* $d^* \in [0, \infty)$, *which maximizes the joint survival probability* $J_S(d)$ *on* $[0, \infty)$, *exists if and only if the equation*

$$d + (1 + \theta_R) \int_d^\infty S(x)dx = P_0 + u_I \tag{2}$$

has solutions in $d \in [0, \infty)$. *Moreover, when the optimal retention* d^* *exists,* d^* *is just the solution to equation* (2) *and satisfies* $J_S(d^*) = \max_{d\in[0,\infty)} J_S(d) = F\big(u_I + u_R + P_0\big)$. □

Throughout this paper, we define

$$\alpha_R = \frac{1}{1 + \theta_R} \quad \text{and} \quad d_R = S^{-1}\Big(\frac{1}{1 + \theta_R}\Big).$$

Remark 2. In the stop-loss reinsurance, we denote the survival probabilities of the insurer and reinsurer by $S_I(d) = \Pr\{X \wedge d \le P_I(d) + u_I\}$

and $S_R(d) = \Pr\{(X-d)_+ \le P_R(d) + u_R\}$, respectively. Thus,

$$S_I(d) = \begin{cases} 1 & \text{if} \quad d \le P_I(d) + u_I, \\ F\big(u_I + P_I(d)\big) & \text{if} \quad d > P_I(d) + u_I, \end{cases}$$

and $S_R(d) = F\big(u_R + d + P_R(d)\big)$.

It is easy to check that under the optimal stop-loss retention d^* given by (2), $d^* = P_I(d^*) + u_I$ and hence $S_I(d^*) = 1$ while $S_R(d^*) = F\big(u_I + u_R + P_0\big)$.

Thus, at the optimal stop-loss retention level d^*, the insurer will survive with a certainty while the reinsurer has a risk of bankrupt. In this sense, a stop-loss reinsurance is not fair to a reinsurer. $\square$

We then determine the optimal stop-loss retention $d^* \in [0, \infty)$, which maximizes the joint profitable probability and satisfies $J_P(d^*) = \sup_{d\in[0,\infty)} J_P(d)$.

Theorem 3. *The optimal stop-loss retention* $d^* \in [0, \infty)$, *which maximizes the joint profitable probability* $J_P(d)$ *on* $[0, \infty)$, *exists if and only if*

$$d_R + (1 + \theta_R) \int_{d_R}^{\infty} S(x) \mathrm{d}x \leq P_0 \tag{3}$$

holds. Moreover, when the optimal stop-loss retention d^* *exists,* d^* *is just the solution to equation*

$$d + (1 + \theta_R) \int_{d}^{\infty} S(x) \mathrm{d}x = P_0 \tag{4}$$

and satisfies $J_P(d^*) = \max_{d \in [0,\infty)} J_P(d) = F\big(P_0\big)$. □

Remark 3. In the stop-loss reinsurance, we denote the profitable probabilities of the insurer and reinsurer by $PR_I(d) = \Pr\{X \wedge d \le P_I(d)\}$ and $PR_R(d) = \Pr\{(X-d)_+ \le P_R(d)\}$, respectively. Thus,

$$PR_I(d) = \begin{cases} 1 & \text{if} \quad d + P_R(d) \le P_0, \\ F\big(P_0 - P_R(d)\big) & \text{if} \quad d + P_R(d) > P_0, \end{cases}$$

and $PR_R(d) = F\big(d + P_R(d)\big)$.

Under the optimal stop-loss retention d^* given by (4), $PR_I(d^*) = 1$, and $PR_R(d^*) = F\big(P_0\big)$. Thus, at the optimal retention level d^*, the insurer will make risk-free profits while the reinsurer has a risk of losing money. In this sense, a stop-loss reinsurance is not fair to a reinsurer. □

4 Optimal reinsurance treaties under general premium principles and among a wide class of reinsurance policies

In the following theorem, we give conditions on $f^* \in \mathcal{F}^\pi$ so that

$$\begin{aligned} J_S^{f^*} &= \Pr\{I_{f^*}(X) \leq P_I^{f^*} + u_I,\ f^*(X) \leq P_R^{f^*} + u_R\} \\ &= \max_{f \in \mathcal{F}^\pi} \Pr\{I_f(X) \leq P_I^f + u_I,\ f(X) \leq P_R^f + u_R\} \\ &= \max_{f \in \mathcal{F}^\pi} J_S^f \end{aligned} \tag{5}$$

holds.

Theorem 4. *If a ceded loss function $f^* \in \mathcal{F}^\pi$ satisfies $f^*(x)$ and $I_{f^*}(x)$ are both non-decreasing functions in $x \geq 0$, and*

$$P_R^{f^*} + u_R = f^*(u_I + u_R + P_0), \tag{6}$$

then f^* is an optimal ceded loss function in $\mathcal{F}^\pi$, which maximizes the joint survival probability and satisfies (5).

Proof: For any $f \in \mathcal{F}^\pi$, it holds that

$$\begin{aligned} J_S^f &= \Pr\{I_f(X) \leq u_I + P_I^f,\, f(X) \leq u_R + P_R^f\} \\ &\leq \Pr\{I_f(X) + f(X) \leq u_I + u_R + P_I^f + P_R^f\} \\ &= \Pr\{X \leq u_I + u_R + P_0\} = F(u_I + u_R + P_0). \end{aligned} \tag{7}$$

If there exists a ceded loss function $f^* \in \mathcal{F}^\pi$ satisfying (6), then it follows from $I_f(x) = x - f(x)$ that

$$\begin{aligned} I_{f^*}(u_I + u_R + P_0) &= u_I + u_R + P_0 - f^*(u_I + u_R + P_0) \\ &= u_I + u_R + P_0 - P_R^{f^*} - u_R = u_I + P_I^{f^*}. \end{aligned}$$

Thus

$$P_I^{f^*} = I_{f^*}(u_I + u_R + P_0) - u_I \quad \text{and} \quad P_R^{f^*} = f^*(u_I + u_R + P_0) - u_R.$$

Hence,

$$\begin{aligned} & \{I_{f^*}(X) \le P_I^{f^*} + u_I,\ f^*(X) \le P_R^{f^*} + u_R\} \\ = \ & \{I_{f^*}(X) \le I_{f^*}(u_I + u_R + P_0),\ f^*(X) \le f^*(u_I + u_R + P_0)\}. \end{aligned}$$

Since $f^*(\cdot)$ and $I_{f^*}(\cdot)$ are both non-decreasing functions, we have

$$\{X \le u_I + u_R + P_0\} \subseteq \{I_{f^*}(X) \le I_{f^*}(u_I + u_R + P_0)\}$$

and

$$\{X \leq u_I + u_R + P_0\} \subseteq \{f^*(X) \leq f^*(u_I + u_R + P_0)\}.$$

One can obtain

$$\begin{aligned}&\{X \leq u_I + u_R + P_0\} \subseteq \\ &\{I_{f^*}(X) \leq I_{f^*}(u_I + u_R + P_0),\ f^*(X) \leq f^*(u_I + u_R + P_0)\},\end{aligned}$$

which means that

$$\begin{aligned}&\Pr\{I_{f^*}(X) \leq I_{f^*}(u_I + u_R + P_0),\ f^*(X) \leq f^*(u_I + u_R + P_0)\} \\ &\geq F(u_I + u_R + P_0).\end{aligned}$$

Therefore, by (7), we have that

$$\Pr\{I_{f^*}(X) \le u_I + P_I^{f^*},\, f^*(X) \le u_R + P_R^{f^*}\} = F(u_I + u_R + P_0) = \max_{f\in\mathcal{F}^\pi} J_S^f.$$

Hence, f^* is the optimal ceded loss function in $\mathcal{F}^\pi$. □

Moreover, in the following theorem, we give conditions on $f^* \in \mathcal{F}^\pi$ so that

$$\begin{aligned} J_P^{f^*} &= \Pr\{I_{f^*}(X) \le P_I^{f^*},\, f^*(X) \le P_R^{f^*}\} \\ &= \max_{f\in\mathcal{F}^\pi} \Pr\{I_f(X) \le P_I^f,\, f(X) \le P_R^f\} = \max_{f\in\mathcal{F}^\pi} J_P^f \end{aligned} \tag{8}$$

holds.

Theorem 5. *If a ceded loss function $f^* \in \mathcal{F}^\pi$ satisfies $f^*(x)$ and $I_{f^*}(x)$ are both non-decreasing functions in $x \geq 0$, and*

$$P_R^{f^*} = f^*(P_0), \tag{9}$$

then f^* is an optimal ceded loss function in $\mathcal{F}^\pi$, which maximizes the joint profitable probability and satisfies (8). □

Remark 4. Theorems 4 and 5 enable one to design optimal reinsurance contracts that maximize the joint survival probability and the joint profitable probability under a general reinsurance premium principle π and among the wide class of reinsurance contracts $\mathcal{F}^\pi$.

We illustrate the applications of Theorems 4 and 5 by designing an optimal reinsurance in the form of a limited stop-loss reinsurance under the expected value principle. □

In a limited stop-loss reinsurance, $f(X) = (X - d_1)_+ \wedge d_2$ and $I_f(X) = X - (X - d_1)_+ \wedge d_2$. The retention for the limited stop-loss reinsurance is the vector (d_1, d_2), where $(d_1, d_2) \in [0, \infty) \times [0, \infty)$.

Let Γ be the set of admissible retentions for the limited stop-loss reinsurance. Then

$$\Gamma = \left\{(d_1, d_2) : \; d_1 \geq 0 \text{ and } d_2 \geq 0\right\}.$$

In this subsection, we first design an optimal limited stop-loss reinsurance $f^*(X) = (X - d_1^*)_+ \wedge d_2^*$, which maximizes the joint profitable probability and satisfies

$$
\begin{aligned}
J_S(d_1^*, d_2^*) &= \max_{(d_1,d_2)\in\Gamma} J_S(d_1, d_2) \\
&= \max_{f\in\mathcal{F}^\pi} \Pr\{I_f(X) \leq P_I^f + u_I,\ f(X) \leq P_R^f + u_R\}. \quad (10)
\end{aligned}
$$

Note that in the limited stop-loss reinsurance and under the expected value principle, the reinsurance premium and the net insurance premium are

$$\begin{aligned} P_R(d_1, d_2) &= (1+\theta_R)\mathrm{E}\big[(X-d_1)_+ \wedge d_2\big] = (1+\theta_R)\int_{d_1}^{d_1+d_2} S(x)\mathrm{d}x, \\ P_I(d_1, d_2) &= P_0 - (1+\theta_R)\int_{d_1}^{d_1+d_2} S(x)\mathrm{d}x. \end{aligned}$$

Throughout this paper, we define

$$\begin{aligned} \Gamma_1 &= \{(d_1, d_2) : (d_1, d_2) \in \Gamma,\ d_1 < u_I + u_R + P_0,\ d_1 + d_2 > u_I + u_R + P_0\}, \\ \Gamma_2 &= \{(d_1, d_2) : (d_1, d_2) \in \Gamma,\ d_1 < u_I + u_R + P_0,\ d_1 + d_2 \le u_I + u_R + P_0\}. \end{aligned}$$

Theorem 6. *Let π be the expected value principle. If the equation*

$$d_1 + (1+\theta_R)\int_{d_1}^{d_1+d_2} S(x)dx = u_I + P_0 \tag{11}$$

has solutions in Γ_1 or the equation

$$d_2 - (1+\theta_R)\int_{d_1}^{d_1+d_2} S(x)dx = u_R \tag{12}$$

has solutions in Γ_2, then a limited stop-loss reinsurance with retention $(d_1^*, d_2^*) \in \Gamma_1^* \cup \Gamma_2^*$ is an optimal reinsurance in $\mathcal{F}^\pi$, which maximizes the joint survival probability and satisfies (10). Here, Γ_1^* and Γ_2^* are the sets of solutions to equations (11) and (12), respectively.

Proof: For the limited stop-loss reinsurance, equation (6) is reduced to

$$(1+\theta_R)\int_{d_1}^{d_1+d_2} S(x)dx + u_R = (u_I + u_R + P_0 - d_1)_+ \wedge d_2. \tag{13}$$

Note that equation (13) has solutions in Γ is equivalent to that equation (11) has solutions in Γ_1 or equation (12) has solutions in Γ_2. Thus, we obtain Theorem 6 by Theorem 4. □

Remark 5. In the limited stop-loss reinsurance, we denote the survival probabilities of the insurer and reinsurer by $S_I(d_1, d_2) = \Pr\{X - (X - d_1)_+ \wedge d_2 \le P_I(d_1, d_2) + u_I\}$ and $S_R(d_1, d_2) = \Pr\{(X - d_1)_+ \wedge d_2 \le P_R(d_1, d_2) + u_R\}$, respectively. Thus,

$$S_I(d_1, d_2) = \begin{cases} F(u_I + P_I(d_1, d_2)) & \text{if} \quad d_1 > u_I + P_I(d_1, d_2), \\ F\big(u_I + d_2 + P_I(d_1, d_2)\big) & \text{if} \quad d_1 \le u_I + P_I(d_1, d_2), \end{cases}$$

and

$$S_R(d_1, d_2) = \begin{cases} F(d_1 + u_R + P_R(d_1, d_2)) & \text{if} \quad d_2 > u_R + P_R(d_1, d_2), \\ 1 & \text{if} \quad d_2 \leq u_R + P_R(d_1, d_2). \end{cases}$$

Under the optimal retentions (d_1^*, d_2^*) given by (11) in Γ_1, it is easy to see that $S_R(d_1^*, d_2^*) = F(u_I + u_R + P_0) < S_I(d_1^*, d_2^*) < 1$. While under the optimal retentions (d_1^*, d_2^*) given by (12) in Γ_2, $S_I(d_1^*, d_2^*) = F(u_I + u_R + P_0)$ and $S_R(d_1^*, d_2^*) = 1$.

It is interesting to note that under the optimal retentions (d_1^*, d_2^*) given by (12) in Γ_2, the reinsurer will survive with a certainty while the insurer has a risk of bankrupt. Hence, the limited stop-loss reinsurance with the retentions (d_1^*, d_2^*) given by (12) in Γ_2 is not fair to the insurer. However, the limited stop-loss reinsurance with the retentions (d_1^*, d_2^*) given by (11) in Γ_1 can avoid such an unfair situation. □

We then design an optimal limited stop-loss reinsurance $f^*(X) = (X - d_1^*)_+ \wedge d_2^*$, which maximizes the joint profitable probability and satisfies

$$J_P(d_1^*, d_2^*) = \max_{(d_1,d_2)\in\Gamma} J_P(d_1, d_2) = \max_{f\in\mathcal{F}^\pi} \Pr\{I_f(X) \leq P_I^f, f(X) \leq P_R^f\}. \quad (14)$$

In this subsection, we define

$$\begin{aligned}
\bar{\Gamma}_1 &= \{(d_1, d_2) : (d_1, d_2) \in \Gamma \ \ d_1 < P_0,\ d_2 > 0,\ d_1 + d_2 > P_0\}, \\
\bar{\Gamma}_2 &= \{(d_1, d_2) : (d_1, d_2) \in \Gamma,\ d_1 < P_0,\ d_2 > 0,\ d_1 + d_2 \leq P_0\}, \\
\bar{\Gamma}_3 &= \{(d_1, d_2) : (d_1, d_2) \in \Gamma,\ d_2 = 0\}.
\end{aligned}$$

Theorem 7. *Let π be the expected value principle. A limited stop-loss reinsurance with retention (d_1^*, d_2^*) is an optimal reinsurance in $\mathcal{F}^\pi$, which maximize the joint profitable probability and satisfies (14), if $(d_1^*, d_2^*) \in \bar{\Gamma}_1$ and satisfies equation*

$$d_1 + (1+\theta_R)\int_{d_1}^{d_1+d_2} S(x)dx = P_0 \tag{15}$$

or $(d_1^*, d_2^*) \in \bar{\Gamma}_2$ and satisfies the equation

$$d_2 - (1+\theta_R)\int_{d_1}^{d_1+d_2} S(x)dx = 0 \tag{16}$$

or $(d_1^*, d_2^*) \in \bar{\Gamma}_3$.

Proof: For the limited stop-loss reinsurance, equation (9) is reduced to

$$(1+\theta_R)\int_{d_1}^{d_1+d_2} S(x)dx = (P_0 - d_1)_+ \wedge d_2. \tag{17}$$

Note that any $(d_1, d_2) \in \bar{\Gamma}_3$ is a solution of equation (17) and equation (17) has solutions in $\bar{\Gamma} = \{(d_1, d_2) : (d_1, d_2) \in \Gamma,\ d_2 > 0\}$ is equivalent to that equation (15) has solutions in $\bar{\Gamma}_1$ or equation (16) has solutions in $\bar{\Gamma}_2$. Thus, we obtain Theorem 7 by Theorem 5. □

Remark 6. In the limited stop-loss reinsurance, we denote the profitable probabilities of the insurer and reinsurer by $PR_I(d_1, d_2) = \Pr\{X - (X - d_1)_+ \wedge d_2 \le P_I(d_1, d_2)\}$ and $PR_R(d_1, d_2) = \Pr\{(X - d_1)_+ \wedge d_2 \le$

$P_R(d_1, d_2)\}$, respectively. Thus,

$$PR_I(d_1, d_2) = \begin{cases} F(P_I(d_1, d_2)) & \text{if} \quad d_1 > P_I(d_1, d_2), \\ F\big(d_2 + P_I(d_1, d_2)\big) & \text{if} \quad d_1 \leq P_I(d_1, d_2), \end{cases}$$

and

$$PR_R(d_1, d_2) = \begin{cases} F(d_1 + P_R(d_1, d_2)) & \text{if} \quad d_2 > P_R(d_1, d_2), \\ 1 & \text{if} \quad d_2 \leq P_R(d_1, d_2). \end{cases}$$

Under the optimal retentions (d_1^*, d_2^*) given by (16) in $\bar{\Gamma}_2$ or $(d_1^*, d_2^*) \in \bar{\Gamma}_3$, $PR_I(d_1^*, d_2^*) = F(P_0)$ and $PR_R(d_1^*, d_2^*) = 1$. While under the optimal retentions (d_1^*, d_2^*) given by (15) in $\bar{\Gamma}_1$, $PR_R(d_1^*, d_2^*) = F(P_0) < PR_I(d_1^*, d_2^*) < 1$.

Note that under the optimal retentions (d_1^*, d_2^*) given by (16) in $\bar{\Gamma}_2$ or $(d_1^*, d_2^*) \in \bar{\Gamma}_3$, the reinsurer will make risk-free profits while the insurer has a risk of losing money.

Hence, the limited stop-loss reinsurance with the retentions (d_1^*, d_2^*) given by (16) in $\bar{\Gamma}_2$ or $(d_1^*, d_2^*) \in \bar{\Gamma}_3$ is not fair to the insurer.

However, the limited stop-loss reinsurance with the retentions (d_1^*, d_2^*) given by (15) in $\bar{\Gamma}_1$ can avoid such an unfair situation. $\square$

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