On the (in-)dependence between financial and actuarial risks under physical and pricing measures

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Agenda

- **Part I**:[1]
  - (In-)dependence under $\mathbb{P}$ versus (in-)dependence under $\mathbb{Q}$.
    - $\mathbb{Q}$ = *a* pricing measure.
- **Part II**:[2]
  - (In-)dependence under $\mathbb{P}$ versus (in-)dependence under $\widehat{\mathbb{Q}}$.
    - $\widehat{\mathbb{Q}}$ = *the* minimal entropy martingale measure.

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[2] **Dhaene, Stassen, Vellekoop, Devolder (2013)**. The Minimal Entropy Martingale Measure in a combined financial-actuarial world. *Work in progress*.

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  - Transfer of underwriting risk to capital markets,
  - through issuance of financial securities,
  - with payoffs contingent on the outcome of quantities related to this underwriting risk.
- Examples:
  - Catastrophe bonds.
  - Longevity bonds.
- Modeling and pricing insurance-linked securities:
  - Financial and actuarial risks.
  - Dependence between financial and actuarial risks.

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Probabilities

- The combined financial-actuarial world:

$$\left(\Omega, \mathcal{F}, \left(\mathcal{F}_t\right)_{0\leq t\leq T}\right)$$

- Consider a market of tradable assets in this combined world.
- Assume that this market is arbitrage-free.
- Physical probability measure $\mathbb{P}$:
  - Used for probability statements about future evolutions of financial and actuarial risks.
- Pricing probability measure $\mathbb{Q}$:
  - Used for expressing prices of tradable assets.
  - Price recipy:
    The current price $S(0)$ of a traded asset with pay-off $S(T)$ at time T can be expressed as:

$$\boxed{S(0) = e^{-rT}\ \mathbb{E}^{\mathbb{Q}}\left[S(T)\right]}$$

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Introduction

A Black & Scholes - setting

- A correlated Brownian motion process:
  - $\left\{\left(B^{(1)}(t), B^{(2)}(t)\right) \mid 0 \leq t \leq T\right\}$,
  - defined on $\left(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P}\right)$,
  - $(\mathcal{F}_t)_{0 \leq t \leq T}$ is the 'natural filtration' induced by this process,
  - $\text{Corr}_{\mathbb{P}}\left[B^{(1)}(t), B^{(2)}(t)\right] = \rho$.
- A market of tradable assets:
  a deterministic risk-free interest rate r,
  a financial asset$^{(1)}$ and an actuarial asset$^{(2)}$.
- $\mathbb{P}$-dynamics of asset prices:

$$\frac{dS^{(i)}(t)}{S^{(i)}(t)} = \mu^{(i)} dt + \sigma^{(i)} dB^{(i)}(t), \qquad i = 1, 2.$$

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- $\mathbb{Q}$-dynamics of asset prices:

$$\frac{dS^{(i)}(t)}{S^{(i)}(t)} = r\ dt + \sigma^{(i)} dW^{(i)}(t)\,, \qquad i = 1, 2.$$

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- Consider the asset prices $S^{(1)}(t)$ and $S^{(2)}(t)$.
  - $\mathbb{P}$ - independence $\Leftrightarrow$ $\mathbb{Q}$ - independence.
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A general setting

- For a general asset pricing model:
  - $\mathbb{P}$ - copula $\neq$ $\mathbb{Q}$ - copula.
  - $\mathbb{P}$ - independence $\nLeftrightarrow$ $\mathbb{Q}$ - independence.
  - $\mathbb{P}$ - comonotonicity $\Leftrightarrow$ $\mathbb{Q}$ - comonotonicity.
- Assuming independence between financial and actuarial risks:
  - $\mathbb{P}$ - independence might be a *reasonable* assumption.
  - $\mathbb{Q}$ - independence is a *convenient* assumption.
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A (simple) combined financial-biometrical world

- A simple world: $\boxed{(\Omega, \mathcal{F}, \mathbb{P})}$
  - Single period, finite state setting.
  - We consider time 0 (= now) and time 1.
- Risks = r.v.'s of which outcome is known at time 1:
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The financial world

- Risk-free bond:
  - Interest rate $r = 0$.
- Stock:
  - Current price: $S^{(1)}(0) = 100$.
  - Price at time 1: $S^{(1)}(1)$, which is either 50 or 150.
- **Financial world**:

$$\boxed{\left(\Omega^{(1)}, \mathcal{F}^{(1)}, \mathbb{P}^{(1)}\right)}$$

  - Universe:

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Equivalent martingale measures

- Supppose that the global world $(\Omega, \mathcal{F}, \mathbb{P})$ is home to a market of tradable assets.
- $\mathbb{Q}$ is an **equivalent martingale measure** if:
  - $\mathbb{Q}$ is a probability measure on $(\Omega, \mathcal{F})$.
  - $\mathbb{Q}$ and $\mathbb{P}$ are equivalent.
  - The current price $S(0)$ of any tradable asset with pay-off $S(1)$ at time 1 can be expressed as

$$\boxed{S(0) = \mathbb{E}^{\mathbb{Q}}\left[S(1)\right]}$$

- **Fundamental Theorems of Asset Pricing**:
  - The market is *arbitrage-free* if and only if there exists an equivalent martingale measure $\mathbb{Q}$.
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- **Fundamental Theorems of Asset Pricing**:
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A combined financial-biometrical world

Equivalent martingale measures

- Supppose that the global world $(\Omega, \mathcal{F}, \mathbb{P})$ is home to a market of tradable assets.
- $\mathbb{Q}$ is an **equivalent martingale measure** if:
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A combined financial-biometrical world

Equivalent martingale measures

- Projection of $\mathbb{Q}$ on the financial world: $\mathbb{Q}^{(1)}$.

$$\begin{cases} \mathbb{Q}^{(1)}[50] = \mathbb{Q}[50, 0] + \mathbb{Q}[50, 1] \\ \mathbb{Q}^{(1)}[150] = \mathbb{Q}[150, 0] + \mathbb{Q}[150, 1] \end{cases}$$

- Projection of $\mathbb{Q}$ on the biometrical world: $\mathbb{Q}^{(2)}$.
- The product measure $\mathbb{Q}^{(1)} \times \mathbb{Q}^{(2)}$:

$$\left(\mathbb{Q}^{(1)} \times \mathbb{Q}^{(2)}\right)[\omega_1, \omega_2] = \mathbb{Q}^{(1)}[\omega_1] \times \mathbb{Q}^{(2)}[\omega_2]$$

- Financial and biometrical risks are independent under $\mathbb{Q}$ if

$$\boxed{\mathbb{Q} = \mathbb{Q}^{(1)} \times \mathbb{Q}^{(2)}}$$

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A combined financial-biometrical world

An incomplete market with 2 purely financial securities

- **Traded securities**:
  - Risk-free bond: $r = 0$.
  - Stock:
    - Current price: $S^{(1)}(0) = 100$.
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- Determining $\mathbb{Q}$: Find positive $\mathbb{Q}[50, 0], \ldots$ satisfying

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A combined financial-biometrical world

An incomplete market with 2 purely financial securities

- Two particular pricing measures:

$$\left\{ \begin{array}{ll} \overline{\mathbb{Q}}\,[50,0] & = 0.2 \\ \overline{\mathbb{Q}}\,[150,0] & = 0.1 \\ \overline{\mathbb{Q}}\,[50,1] & = 0.3 \\ \overline{\mathbb{Q}}\,[150,1] & = 0.4 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ll} \overline{\mathbb{Q}}^{(1)} \times \overline{\mathbb{Q}}^{(2)}\,[50,0] & = 0.15 \\ \overline{\mathbb{Q}}^{(1)} \times \overline{\mathbb{Q}}^{(2)}\,[150,0] & = 0.15 \\ \overline{\mathbb{Q}}^{(1)} \times \overline{\mathbb{Q}}^{(2)}\,[50,1] & = 0.35 \\ \overline{\mathbb{Q}}^{(1)} \times \overline{\mathbb{Q}}^{(2)}\,[150,1] & = 0.35 \end{array} \right.$$

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A combined financial-biometrical world

An incomplete market with 2 purely financial and 1 purely biometrical security

- **Traded securities**:
  - Risk-free bond: $r = 0$.
  - Stock: $S^{(1)}$.
  - Biometrical security:
    - Current price: $S^{(2)}(0) = 70$.
    - Pay-off at time 1:

$$S^{(2)}(1) = 100 \times \mathcal{I}(1)$$

- Determining $\mathbb{Q}$: Find positive $\mathbb{Q}[50, 0], \ldots$ satisfying

$$\begin{cases} \mathbb{E}^{\mathbb{Q}}\left[S^{(1)}(1)\right] = 100 \\ \mathbb{E}^{\mathbb{Q}}\left[S^{(2)}(1)\right] = 70 \\ \mathbb{Q}[50, 0] + \mathbb{Q}[150, 0] + \mathbb{Q}[50, 1] + \mathbb{Q}[150, 1] = 1 \end{cases}$$

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- Equivalent with: Find positive $\mathbb{Q}[50, 0]$, ... satisfying

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- $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}^{(1)} \times \overline{\mathbb{Q}}^{(2)}$ (defined earlier) are 2 particular pricing measures.
- Conclusions:
  - The market is arbitrage-free but incomplete.
  - Under $\overline{\mathbb{Q}}$, the independence property is not maintained.
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$$\begin{cases} \mathbb{Q}^{(1)}[50] = 0.5 \\ \mathbb{Q}^{(1)}[150] = 0.5 \\ \mathbb{Q}^{(2)}[0] = 0.3 \\ \mathbb{Q}^{(2)}[1] = 0.7 \end{cases}$$

- $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}^{(1)} \times \overline{\mathbb{Q}}^{(2)}$ (defined earlier) are 2 particular pricing measures.
- Conclusions:
  - The market is arbitrage-free but incomplete.
  - Under $\overline{\mathbb{Q}}$, the independence property is not maintained.
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A complete market with 2 financial, 1 biometrical and 1 combined security

- **Traded securities**:
  - Risk-free bond: $r = 0$.
  - Stock: $S^{(1)}$.
  - Biometrical security: $S^{(2)}$.
  - Combined security:
    - Current price: $\mathbf{S(0)} \in (10, 25)$.
    - Pay-off at time 1:

$$S(1) = \left(100 - S^{(1)}(1)\right)_+ \times \mathcal{I}(1)$$

- Determining $\mathbb{Q}$: Find positive $\mathbb{Q}[50, 0], \ldots$ satisfying

$$\begin{cases} \mathbb{E}^{\mathbb{Q}}\left[S^{(1)}(1)\right] = 100 \\ \mathbb{E}^{\mathbb{Q}}\left[S^{(2)}(1)\right] = 70 \\ \mathbb{E}^{\mathbb{Q}}\left[S(1)\right] = S(0) \\ \mathbb{Q}[50, 0] + \mathbb{Q}[150, 0] + \mathbb{Q}[50, 1] + \mathbb{Q}[150, 1] = 1 \end{cases}$$

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- Unique pricing measure $\widetilde{\mathbb{Q}}$:

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  - The market is arbitrage-free and complete.
  - The unique pricing measure maintains the independence property if and only if $S(0) = 17.5$.
  - In case $S(0) \notin (10, 25)$: the market is not arbitrage-free.

- Conclusion:

  - In an arbitrage-free and complete market, it may happen that the unique pricing measure does not maintain the independence property.

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- **Financial world**:

$$\boxed{\left(\Omega^{(1)}, \mathcal{F}^{(1)}, \mathbb{P}^{(1)}\right)}$$

  - Universe:

$$\Omega^{(1)} = \{B, M, R\}$$

    - Booming economy, Moderate growth, Recession.

  - Real-world probabilities:

$$\mathbb{P}^{(1)}[B] > 0,\ \mathbb{P}^{(1)}[M] > 0 \text{ and } \mathbb{P}^{(1)}[R] > 0$$

- **Biometrical world**: $\left(\Omega^{(2)}, \mathcal{F}^{(2)}, \mathbb{P}^{(2)}\right)$ as defined before.

Another (simple) combined financial-biometrical world

The financial and the biometrical world

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The global world

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    Financial and biometrical risks are assumed to be independent:

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An incomplete market with 2 financial, 1 biometrical and 1 combined security

- **Traded securities**:
  - Risk-free bond: $r = 0$.
  - Financial security: Current price: $S^{(1)}(0) = 50$.
    - Pay-off at time 1:

$$S^{(1)}(1) = \begin{cases} 100, & \text{if } B \\ 0, & \text{otherwise} \end{cases}$$

  - Biometrical security: Current price: $S^{(2)}(0) = 70$.
    - Pay-off at time 1:

$$S^{(2)}(1) = 100 \times \mathcal{I}(1)$$

  - Combined security: Current price: $\mathbf{S(0)} \in (0, 30)$.
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Chosing a pricing measure in an incomplete market

- Consider an arbitrage-free market of tradable assets in a combined financial - actuarial world.
- Suppose that this market is incomplete.
  - There exists more than 1 equivalent martingale measure.
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  - Chosing the measure $\widehat{\mathbb{Q}}$ that is *closest* to $\mathbb{P}$.
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The global world

- Consider a **single period, finite state world** $(\Omega, \mathcal{F}, \mathbb{P})$.
- The universe:

$$\Omega = \left\{ (i,j) \mid i = 1, \ldots, n^{(f)} \text{ and } j = 1, \ldots, n^{(a)} \right\},$$

  - Any (i,j) corresponds to a *global state* of the world:
  - $i =$ *financial substate* of the world,
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Pricing assets in the global market

Equivalent martingale measures

- Consider a market of tradable assets in the global world $(\Omega, \mathcal{F}, \mathbb{P})$.
- We assume that this market is arbitrage-free.
- There exists at least 1 equivalent martingale measure $\mathbb{Q}$:

$$\mathbb{Q}\left[(i,j)\right] = q_{ij}$$

- The projections $\mathbb{Q}^{(f)}$ and $\mathbb{Q}^{(a)}$ of $\mathbb{Q}$ to the subworlds:

$$q_i^{(f)} = \sum_{j=1}^{n^{(a)}} q_{ij} = \quad \text{and} \quad q_j^{(a)} = \sum_{i=1}^{n^{(f)}} q_{ij}$$

- $\mathbb{Q}^{(f)}$ and $\mathbb{Q}^{(a)}$ are equivalent martingale measures for the respective submarkets in the subworlds.

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Independence between financial and actuarial risks

- The probability measure $\mathbb{P}^{(f)} \times \mathbb{P}^{(a)}$:

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- $\mathbb{P}$-independence:

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The Minimal Entropy Martingale Measure

Relative entropy (Kulback-Leibner information)

- Consider the probability measures $\mathbb{P}$ and $\mathbb{Q}$ defined on $(\Omega, \mathcal{F})$.
- Relative entropy of $\mathbb{Q}$ wrt $\mathbb{P}$:

$$I\left(\mathbb{Q}, \mathbb{P}\right) = \sum_{i,j} q_{ij} \ln\left(\frac{q_{ij}}{p_{ij}}\right)$$

  - sum over all i, j with $p_{ij} > 0$.
  - $0 \ln 0 = 0$ by convention.
- Properties:
  - $I\left(\mathbb{Q}, \mathbb{P}\right) \geq 0$.
  - $I\left(\mathbb{Q}, \mathbb{P}\right) = 0 \Leftrightarrow \mathbb{Q} \equiv \mathbb{P}$.
  - $I\left(\mathbb{Q}, \mathbb{P}\right)$ increases if the measures 'diverge'.
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- Consider the probability measures $\mathbb{P}$ and $\mathbb{Q}$ defined on $(\Omega, \mathcal{F})$.
- Relative entropy of $\mathbb{Q}$ wrt $\mathbb{P}$:

$$\boxed{I\left(\mathbb{Q}, \mathbb{P}\right) = \sum_{i,j} q_{ij} \ln\left(\frac{q_{ij}}{p_{ij}}\right)}$$

  - sum over all i, j with $p_{ij} > 0$.
  - $0 \ln 0 = 0$ by convention.
- Properties:
  - $I\left(\mathbb{Q}, \mathbb{P}\right) \geq 0$.
  - $I\left(\mathbb{Q}, \mathbb{P}\right) = 0 \Leftrightarrow \mathbb{Q} \equiv \mathbb{P}$.
  - $I\left(\mathbb{Q}, \mathbb{P}\right)$ increases if the measures 'diverge'.
  - $I\left(\mathbb{Q}, \mathbb{P}\right)$ is not a 'distance' in the strict sense.

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The global entropy measure

- Consider the global world $(\Omega, \mathcal{F}, \mathbb{P})$.
- This world is home to a market of tradable assets.
- $\mathcal{M}$ = the (non-empty) set of all martingale measures.
- The **Minimal Entropy Martingale Measure** $\widehat{\mathbb{Q}} \in \mathcal{M}$ satisfies[4]:

$$\boxed{I\left(\widehat{\mathbb{Q}}, \mathbb{P}\right) = \min_{\mathbb{Q} \in \mathcal{M}} I(\mathbb{Q}, \mathbb{P})}$$

- We will call $\widehat{\mathbb{Q}}$ the 'global entropy measure'.
- The global entropy measure always exists and is unique.

[4] Frittelli (1995, 2000)

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A market with only purely financial assets

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- Consider the global world $(\Omega, \mathcal{F}, \mathbb{P})$ and the following **market of tradable assets**:
  - A *risk-free bond* (interest rate r).
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$$\widehat{q}_{ij} = p_{ij} \times \frac{\exp\left(\widehat{\lambda}\ s_i^{(f)}\right)}{\mathbb{E}^{\mathbb{P}}\left[\exp\left(\widehat{\lambda}\ S^{(f)}(1)\right)\right]}$$

  - $\widehat{\lambda}$ is the unique solution of

$$\sum_i p_i^{(f)} \left(s_i^{(f)} - e^r\ s_0^{(f)}\right) \exp\left(\lambda\ s_i^{(f)}\right) = 0$$

  - $\widehat{\mathbb{Q}}$ is *equivalent* to $\mathbb{P}$.
  - $\widehat{\mathbb{Q}}$ is an *Esscher transform* of $\mathbb{P}$.

- Projection of $\widehat{\mathbb{Q}}$ to the financial subworld:

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The financial subworld

- Consider the financial subworld $\left(\Omega^{(f)}, \mathcal{F}^{(f)}, \mathbb{P}^{(f)}\right)$ and the corresponding financial submarket.
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- Relation between the entropy measures $\widetilde{\mathbb{Q}}^{(f)}$ and $\widehat{\mathbb{Q}}$:

$$\boxed{\widetilde{\mathbb{Q}}^{(f)} = \widehat{\mathbb{Q}}^{(f)}}$$

  - The financial entropy measure is identical to the projection of the global entropy measure on the financial subworld.

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The actuarial subworld

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- Solution:

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A market with only purely financial assets

Theorem

- Consider the global world $(\Omega, \mathcal{F}, \mathbb{P})$ which is home to a market where only a risk-free bond and a purely financial asset are traded.
- $\widehat{\mathbb{Q}}^{(f)}$ and $\widehat{\mathbb{Q}}^{(a)}$: projections of the global entropy measure $\widehat{\mathbb{Q}}$.
- $\widetilde{\mathbb{Q}}^{(f)}$ and $\widetilde{\mathbb{Q}}^{(a)} \equiv \mathbb{P}^{(a)}$: financial and actuarial entropy measures.
- Then:

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A market with only purely financial assets

Theorem

- Consider the global world $(\Omega, \mathcal{F}, \mathbb{P})$ which is home to a market where only a risk-free bond and a purely financial asset are traded.
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A market with purely financial and purely actuarial assets

The global world

- Consider the global world $(\Omega, \mathcal{F}, \mathbb{P})$ and the following **market of tradable assets**:
  - A *risk-free bond* (interest rate r).
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- The class $\mathcal{M}$ = all probability measures $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ satisfying

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The global world

- <u>The global entropy measure $\widehat{\mathbb{Q}}$</u>:

$$\boxed{I\left(\widehat{\mathbb{Q}},\mathbb{P}\right)=\min_{\mathbb{Q}\in\mathcal{M}} I\left(\mathbb{Q},\mathbb{P}\right)}$$

- <u>Solution</u>:

$$\boxed{\widehat{q}_{ij}=p_{ij}\times\frac{\exp\left(\widehat{\lambda}^{(f)}s_i^{(f)}+\widehat{\lambda}^{(a)}s_j^{(a)}\right)}{\mathbb{E}^{\mathbb{P}}\left[\exp\left(\widehat{\lambda}^{(f)}S^{(f)}(1)+\widehat{\lambda}^{(a)}S^{(a)}(1)\right)\right]}}$$

- <u>$\widehat{\lambda}^{(f)}$ and $\widehat{\lambda}^{(a)}$ follow from</u>:

$$\begin{cases}\displaystyle\sum_{i,j} p_{ij}\times\left(s_i^{(f)}-e^r\ s_0^{(f)}\right)\exp\left(\lambda^{(f)}s_i^{(f)}+\lambda^{(a)}s_j^{(a)}\right)=0\\ \displaystyle\sum_{i,j} p_{ij}\times\left(s_j^{(a)}-e^r\ s_0^{(a)}\right)\exp\left(\lambda^{(f)}s_i^{(f)}+\lambda^{(a)}s_j^{(a)}\right)=0\end{cases}$$

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A market with purely financial and purely actuarial assets

The subworlds

- Consider the actuarial subworld $\left(\Omega^{(a)}, \mathcal{F}^{(a)}, \mathbb{P}^{(a)}\right)$ and the corresponding submarket.
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- In general:

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- $\mathbb{P}$-independence between financial and actuarial risks does not imply $\mathbb{Q}$ - independence.
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Further research

- Dependency structure conserving conditions:

$$\text{When does } \mathrm{PQD}_{\mathbb{P}}\left[X, Y\right] \text{ imply } \mathrm{PQD}_{\mathbb{Q}}\left[X, Y\right] \text{ ?}$$

- Stochastic order conserving conditions:

$$\text{When does } X \leq_{\mathbb{P}-cx} Y \text{ imply } X \leq_{\mathbb{Q}-cx} Y \text{ ?}$$

- Fair valuation of insurance liabilities:

$$\widehat{\mathbb{Q}}^{(1)} \times \mathbb{P}^{(2)}$$

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References I

- Dhaene J., Kukusk A., Luciano E., Schoutens W., Stassen B. (2013). On the (in-)dependence between financial and actuarial risks. *Insurance: Mathematics & Economics*, 52(3), 522-531.
- Dhaene J., Stassen B., Vellekoop M., Devolder P. (2013). The Minimal Entropy Martingale Measure in a combined financial-actuarial world. *Work in progress*.
- Frittelli M. (1995). Minimal entropy criterion for pricing in one period incomplete markets. *Working Paper 99, Dip. Met. Quant., University of Brescia, Italy*.
- Frittelli M. (2000). The minimal entropy martingale measure and the valuation problem in incomplete markets. *Mathematical Finance*, 10(1), 39-52.

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