Valuing equity-linked death benefits and exotic contingent options in jump diffusion models

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Equity-linked death benefit

- x age at issue of policy (time 0)
- T_x time of death
- payment at time T_x
- depends on $S(T_x)$,
- or more generally on $S(t)$, $0 \leq t \leq T_x$

Goal:

- Calculate $\mathrm{E}[e^{-\delta T_x} \times \text{payment}]$
- the expectation of the discounted value of the payment

  δ valuation force of interest

Index process

- $S(t)$ price of one unit of a fund at time t
- $S(t) = S(0)e^{X(t)}, \quad t \geq 0$
- $X(t)$ a Lévy process

$\{X(t)\}$ a jump-diffusion process

$$X(t) = \text{Brownian motion } (\mu, D = \sigma^2/2)$$

$$- \text{ compound Poisson } (\omega, q(x))$$

downward jumps

$$+ \text{ compound Poisson } (\nu, p(x))$$

upward jumps

Double exponential jump-diffusion

- $\nu,\ p(x) = ve^{-vx},\ x > 0$ upward
- $\omega,\ q(x) = we^{-wx},\ x > 0$ downward
- a model with 6 parameters
- a reasonable compromise between Brownian motion and a general Lévy process

More general jump-diffusions

- $\nu,\ p(x) = \sum_{i=1}^{n} P_i v_i e^{-v_i x},\ x > 0$ upward
- $\omega,\ q(x) = \sum_{i=1}^{m} Q_i w_i e^{-w_i x},\ x > 0$ downward
- Kou and his co-authors' work
- a reasonable compromise between Brownian motion and a general Lévy process

$\Psi(z)$: the Lévy exponent

- $\mathrm{E}[e^{zX(t)}] = e^{t\Psi(z)}$
- $\Psi(z) =$
  $Dz^2 + \mu z - \nu \int_0^\infty (1 - e^{zx}) p(x) dx - \omega \int_0^\infty (1 - e^{-zx}) q(x) dx$
- For the double exponential jump-diffusion

  $\Psi(z) = Dz^2 + \mu z + \nu \frac{z}{v-z} - \omega \frac{z}{w+z}$

- $\Psi(z) = 0$: Lundberg's equation
- $\Psi(z) = \lambda$: Generalized Lundberg's equation
- For the double exponential jump-diffusion, this equation has "four" solutions

$$-\infty < \alpha_2 < -w < \alpha_1 < 0 < \beta_1 < v < \beta_2 < \infty$$

For Brownian motion

- $Dz^2 + \mu z = \lambda$: Generalized Lundberg's equation
- Two solutions: $\alpha < 0, \ \beta > 0$

Connection with martingales

- $\{e^{-\lambda t} e^{zX(t)}\}$ is a martingale $*$

  $\leftrightarrow$ z solution of the generalized Lundberg's equation

- Alternative formulation of $*$
  τ independent exponential random variable with parameter λ,
  $\Pr(\tau > t) = e^{-\lambda t}$
  $\{I_{(\tau > t)} e^{zX(t)}\}$ martingale

The reduced problem

- Idea: the pdf of T_x can be approximated by

$$\sum_{i=1}^{n} A_i \lambda_i e^{-\lambda_i t}, \qquad t > 0$$

So, it suffices that we know how to calculate

$$\mathrm{E}[e^{-\delta\tau} b(S(\tau))] = \int_0^\infty e^{-\delta t} \lambda e^{-\lambda t} \mathrm{E}[b(S(t))] dt$$

The reduced problem

δ can be eliminated

$$\int_0^\infty e^{-\delta t} \lambda e^{-\lambda t} \mathrm{E}[b(S(t))] dt$$
$$= \frac{\lambda}{\lambda + \delta} \int_0^\infty (\lambda + \delta) e^{-(\lambda+\delta)t} \mathrm{E}[b(S(t))] dt$$

rule: do the calculation without discounting
but replace λ by $\lambda + \delta$ multiply by $\frac{\lambda}{\lambda+\delta}$

Note:

- If λ is replaced by $\lambda + \delta$,

  the generalized Lundberg's equation is

$$\Psi(z) = \lambda + \delta$$

- Thus the α's and β's are modified accordingly

- Want to calculate

$$E[e^{-\delta\tau} b(S(\tau))] = E[e^{-\delta\tau} b(S(0)e^{X(\tau)})]$$

- so we need

$f_{X(\tau)}(x)$ the pdf of $X(\tau)$

Distribution of $X(\tau)$

$$\mathsf{E}[e^{zX(\tau)}] = \mathsf{E}[\mathsf{E}[e^{zX(\tau)} \mid \tau]] = \mathsf{E}[e^{\tau\Psi(z)}] = \frac{\lambda}{\lambda - \Psi(z)}$$

$$= \frac{\lambda}{\Psi'(\alpha_1)}\frac{1}{\alpha_1 - z} + \frac{\lambda}{\Psi'(\alpha_2)}\frac{1}{\alpha_2 - z} + \frac{\lambda}{\Psi'(\beta_1)}\frac{1}{\beta_1 - z} + \frac{\lambda}{\Psi'(\beta_2)}\frac{1}{\beta_2 - z}$$

$$\rightarrow \quad f_{X(\tau)}(x) = \begin{cases} a_1 e^{-\alpha_1 x} + a_2 e^{-\alpha_2 x}, & x < 0, \\ b_1 e^{-\beta_1 x} + b_2 e^{-\beta_2 x}, & x \geq 0 \end{cases}$$

with

$$a_j = \frac{-\lambda}{\Psi'(\alpha_j)}, \quad b_j = \frac{\lambda}{\Psi'(\beta_j)}, \qquad j = 1, 2.$$

$b(s) = (K - s)_+$ put option

- out-of-the money: $S(0) \geq K$

$$\mathcal{E}_b(S(0)) = \mathrm{E}[(K - S(\tau))_+] = a_1 \eta(\alpha_1; K, S(0)) + a_2 \eta(\alpha_2; K, (S(0))$$

with

$$\eta(h; K, S(0)) = \frac{K^{1-h} S(0)^h}{(h-1)h}$$

$b(s) = (s - K)_+$ call option

- out-of-the money: $S(0) \leq K$

$$\mathcal{E}_b(S(0)) = b_1 \eta(\beta_1; K, S(0)) + b_2 \eta(\beta_2; K, S(0))$$

with

$$\eta(h; K, S(0)) = \frac{K^{1-h} S(0)^h}{(h-1)h}$$

In-the-money formulas

Use put-call parity

$$[K - S(\tau)]_+ - [S(\tau) - K]_+ = K - S(\tau)$$

$$\mathrm{E}[[K - S(\tau)]_+] - \mathrm{E}[[S(\tau) - K]_+] = K - \mathrm{E}[S(\tau)],$$

Running maximum

- $M(t) = \max\{X(u); 0 \leq u \leq t\}$ running maximum
- For the discussion of lookback call options, we need the distribution of $M(\tau)$
- result:

$$f_{M(\tau)}(x) = \frac{\beta_2(v-\beta_1)}{v(\beta_2-\beta_1)}\beta_1 e^{-\beta_1 x} + \frac{\beta_1(\beta_2-v)}{v(\beta_2-\beta_1)}\beta_2 e^{-\beta_2 x}, \qquad x \geq 0,$$

β_1 and β_2 are the positive solutions of $\Psi(z) = \lambda$

Proof

$$\Pr(M(\tau) \geq x) = \Pi_d(x) + \Pi_s(x),$$

where

$\Pi_d(x)$ is the probability that the process exceeds x before time τ and when it occurs, it is because of oscillation.

$\Pi_s(x)$ is the probability that the process exceeds x before time τ and when it occurs, it is because of an upward jump.

Proof

Stop the martingale $\{e^{\beta_1 X(t)} I_{(t<\tau)}; t \geq 0\}$ the first time when $\{X(t)\}$ attains or jumps over level x. Optional sampling theorem yields

$$1 = e^{\beta_1 x}\Pi_d(x) + \frac{v}{v-\beta_1} e^{\beta_1 x}\Pi_s(x).$$

By analytical analogy we have

$$1 = e^{\beta_2 x}\Pi_d(x) + \frac{v}{v-\beta_2} e^{\beta_2 x}\Pi_s(x).$$

Proof

Solution:

$$\Pi_d(x) = \frac{(v-\beta_1)e^{-\beta_1 x} + (\beta_2 - v)e^{-\beta_2 x}}{\beta_2 - \beta_1},$$
$$\Pi_s(x) = \frac{(v-\beta_1)(\beta_2 - v)(e^{-\beta_1 x} - e^{-\beta_2 x})}{v(\beta_2 - \beta_1)}.$$

Then

$$\Pr(M(\tau) \geq x) = \frac{\beta_2(v-\beta_1)e^{-\beta_1 x} + \beta_1(\beta_2 - v)e^{-\beta_2 x}}{v(\beta_2 - \beta_1)}, \quad x \geq 0,$$

$$f_{M(\tau)}(x) = \frac{\beta_2(v-\beta_1)}{v(\beta_2-\beta_1)}\beta_1 e^{-\beta_1 x} + \frac{\beta_1(\beta_2 - v)}{v(\beta_2-\beta_1)}\beta_2 e^{-\beta_2 x}, \qquad x \geq 0,$$

Running minimum

- $m(t) = \min\{X(u); 0 \leq u \leq t\}$ running minimum
- For the discussion of lookback put options, we need the distribution of $m(\tau)$
- result:

$$f_{m(\tau)}(x) = \frac{-\alpha_2(\alpha_1 + w)}{w(\alpha_1 - \alpha_2)}(-\alpha_1 e^{-\alpha_1 x}) + \frac{\alpha_1(w + \alpha_2)}{w(\alpha_1 - \alpha_2)}(-\alpha_2 e^{-\alpha_2 x}),$$

α_1 and α_2 are the negative solutions of $\Psi(z) = \lambda$

Proof

$$\min\{X(t); 0 \leq t \leq \tau\}$$

$$= -\max\{-X(t); 0 \leq t \leq \tau\}$$

Lookback call options

$$b = [\max\{S(t); 0 \leq t \leq \tau\} - K]_+ = [S(0)e^{M(\tau)} - K]_+$$

out-of-the money: $S(0) \leq K$

$$\int_{\ln[K/S(0)]}^{\infty} [S(0)e^x - K]_+ f_{M(\tau)}(x)\mathrm{d}x$$
$$= \frac{\beta_1\beta_2}{v(\beta_2 - \beta_1)}[(v - \beta_1)\eta(\beta_1; K) + (\beta_2 - v)\eta(\beta_2; K)]$$

Lookback put options

$$b = [K - \min\{S(t); 0 \le t \le \tau\}]_+ = [K - S(0)e^{m(\tau)}]_+$$

out-of-the money: $S(0) \ge K$

$$\int_{-\infty}^{\ln[S(0)/K]} [K - S(0)e^x] f_{m(\tau)}(x) \mathrm{d}x$$
$$= \frac{\alpha_1 \alpha_2}{w(\alpha_1 - \alpha_2)} [(\alpha_1 + w)\eta(\alpha_1; K) - (w + \alpha_2)\eta(\alpha_2; K)]$$

Joint pdf of $X(\tau)$ and $M(\tau)$

By

$$X(\tau) = M(\tau) - [M(\tau) - X(\tau)],$$

and independence of $M(\tau)$ and $M(\tau) - X(\tau)$, the joint pdf of $X(\tau)$ and $M(\tau)$, for $y \geq \max(0, x)$, is

$$\begin{aligned} f_{X(\tau),M(\tau)}(x,y) &= f_{M(\tau),M(\tau)-X(\tau)}(y, y-x) \\ &= f_{M(\tau)}(y)\ f_{M(\tau)-X(\tau)}(y-x) \\ &= \frac{\lambda}{D(\alpha_1 - \alpha_2)(\beta_2 - \beta_1)}\ \{-\}, \end{aligned}$$

with

$$
\begin{aligned}
\{-\} \quad = \quad & (\alpha_1 + w)(v - \beta_1)e^{-\alpha_1 x}e^{(\alpha_1-\beta_1)y} \\
& -(w + \alpha_2)(v - \beta_1)e^{-\alpha_2 x}e^{(\alpha_2-\beta_1)y} \\
& +(\alpha_1 + w)(\beta_2 - v)e^{-\alpha_1 x}e^{(\alpha_1-\beta_2)y} \\
& -(w + \alpha_2)(\beta_2 - v)e^{-\alpha_2 x}e^{(\alpha_2-\beta_2)y}.
\end{aligned}
$$

Single barrier options

- Up-and-in option, $S(0) < L$, notation: $\ell = \ln[L/S(0)]$
- payoff at time τ
  $I_{([\max_{0\leq t\leq \tau} S(t)]\geq L)} b(S(\tau)) = I_{(M(\tau)\geq \ell)} b(S(0)e^{X(\tau)})$

  Expected payoff $\int_{\ell}^{\infty}\left[\int_{-\infty}^{y} b(S(0)e^{x}) f_{X(\tau),M(\tau)}(x,y)\mathrm{d}x\right]\mathrm{d}y$

  Need the joint pdf of $X(\tau)$ and $M(\tau)$

Alternative expression for expected payoff

$$\Pi_d(\ell)\mathcal{E}_b(L) + \Pi_s(\ell)\int_0^\infty \mathcal{E}_b(Le^x)ve^{-vx}\mathrm{d}x,$$

where

$$\begin{aligned}
\Pi_d(\ell) &= \frac{(v-\beta_1)(\frac{S(0)}{L})^{\beta_1} + (\beta_2 - v)(\frac{S(0)}{L})^{\beta_2}}{\beta_2 - \beta_1}, \\
\Pi_s(\ell) &= \frac{(v-\beta_1)(\beta_2 - v)[(\frac{S(0)}{L})^{\beta_1} - (\frac{S(0)}{L})^{\beta_2}]}{v(\beta_2 - \beta_1)}.
\end{aligned}$$

For a particular payoff function $b(s)$, it remains to determine $\mathcal{E}_b(L)$ and $\int_0^\infty \mathcal{E}_b(Le^x)ve^{-vx}\mathrm{d}x$.

Up-and-in put option, $L \geq K$

$$\mathcal{E}_b(L) = a_1\eta(\alpha_1; K, L) + a_2\eta(\alpha_2; K, L),$$

and

$$\int_0^\infty \mathcal{E}_b(Le^x)ve^{-vx}\mathrm{d}x = \frac{v}{v-\alpha_1}a_1\eta(\alpha_1; K, L) + \frac{v}{v-\alpha_2}a_2\eta(\alpha_2; K, L),$$

where

$$\eta(h; K, L) = \frac{K^{1-h}L^h}{(h-1)h},$$

Up-and-in put option, $L < K$

$$\mathcal{E}_b(L) = b_1\eta(\beta_1; K, L) + b_2\eta(\beta_2; K, L) + K - L\mathrm{E}[e^{X(\tau)}],$$

and

$$\int_0^\infty \mathcal{E}_b(Le^x) v e^{-vx} \mathrm{d}x = \frac{v}{v-\beta_1}\left[1 - \left(\frac{L}{K}\right)^{v-\beta_1}\right] b_1\eta(\beta_1; K, L)$$
$$+\frac{v}{v-\beta_2}\left[1 - \left(\frac{L}{K}\right)^{v-\beta_2}\right] b_2\eta(\beta_2; K, L)$$
$$+K - L^v K^{1-v} + L\frac{v}{1-v}\mathrm{E}[e^{X(\tau)}]\left[1 - \left(\frac{L}{K}\right)^{v-1}\right]$$
$$+\frac{v}{v-\alpha_1} a_1\eta(\alpha_1; K, L)\left[\frac{K}{L}\right]^{\alpha_1 - v} + \frac{v}{v-\alpha_2} a_2\eta(\alpha_2; K, L)\left[\frac{K}{L}\right]^{\alpha_2 - v}.$$

Brownian motion model

The expectation of the time-τ payoff of a up-and-in barrier option with barrier L, $L > S(0)$, is

$$\left[\frac{S(0)}{L}\right]^{\beta}\mathcal{E}_b(L)$$

For up-and-out case

$$\mathrm{E}[I_{(M(\tau)<\ell)}b(S(\tau))] = \mathcal{E}_b(S(0)) - \left[\frac{S(0)}{L}\right]^{\beta}\mathcal{E}_b(L).$$

The expectation of the time-τ payoff of a down-and-in barrier option with barrier L, $L > S(0)$, is

$$\left[\frac{L}{S(0)}\right]^{-\alpha} \mathcal{E}_b(L).$$

The up-and-out option formula can also be expressed as

$$\mathcal{E}_g(S(0)) - \left[\frac{S(0)}{L}\right]^{\beta} \mathcal{E}_g(L),$$

where the function g is defined by

$$g(s) = I_{(s<L)} b(s),$$

and

$$\mathcal{E}_g(s) = \mathrm{E}[g(S(\tau))|S(0) = s] = \mathrm{E}[I_{(S(\tau)<L)} b(S(\tau))|S(0) = s].$$

Let $s_2 > s_1 \geq L$. If the initial stock price is s_2, $g(S(\tau)) = 0$ unless the stock price drops to the level s_1 before time τ, the probability of which is $(s_1/s_2)^{-\alpha}$.

$$\mathcal{E}_g(s_2) = \left(\frac{s_1}{s_2}\right)^{-\alpha} \mathcal{E}_g(s_1),$$

or

$$\mathcal{E}_g(s_1) = \left(\frac{s_1}{s_2}\right)^{\alpha} \mathcal{E}_g(s_2).$$

In particular,

$$\mathcal{E}_g(L) = \left[\frac{S(0)}{L}\right]^{\alpha} \mathcal{E}_g\left(\frac{L^2}{S(0)}\right).$$

$$\begin{aligned} \mathrm{E}[I_{(M(\tau)<\ell)}\ b(S(\tau))] &= \mathcal{E}_g(S(0)) - \left[\frac{S(0)}{L}\right]^{\alpha+\beta} \mathcal{E}_g\left(\frac{L^2}{S(0)}\right) \\ &= \mathcal{E}_g(S(0)) - \left[\frac{S(0)}{L}\right]^{-\mu/D} \mathcal{E}_g\left(\frac{L^2}{S(0)}\right). \end{aligned}$$

The result can be generalized to case where the barrier is an exponential function of time, $Le^{\xi t},\ t \geq 0$, ξ being a real constant.

Similar results can be obtained for the double-barrier option.

First exit from an interval

Let $x_1 < x < x_2$ and let

$$\mathcal{T} = \min\{t : x + X(t) \le x_1 \text{ or } x + X(t) \ge x_2\}$$

be the exit time of the process $\{x + X(t)\}$ from the interval (x_1, x_2). We are interested in

$$\begin{aligned} \Pi_1(x) &= \Pr(x + X(\mathcal{T}) \le x_1, \mathcal{T} < \tau), \\ \Pi_2(x) &= \Pr(x + X(\mathcal{T}) \ge x_2, \mathcal{T} < \tau). \end{aligned}$$

Results of this kind are needed in the analysis of double barrier options exercisable at time τ.

Four ways to exit the interval

$$\begin{aligned}
\Pi_{1d}(x) &= \Pr(x + X(\mathcal{T}) = x_1, \mathcal{T} < \tau), \\
\Pi_{1s}(x) &= \Pr(x + X(\mathcal{T}) < x_1, \mathcal{T} < \tau), \\
\Pi_{2d}(x) &= \Pr(x + X(\mathcal{T}) = x_2, \mathcal{T} < \tau), \\
\Pi_{2s}(x) &= \Pr(x + X(\mathcal{T}) > x_2, \mathcal{T} < \tau).
\end{aligned}$$

Then

$$\begin{aligned}
\Pi_1(x) &= \Pi_{1d}(x) + \Pi_{1s}(x), \\
\Pi_2(x) &= \Pi_{2d}(x) + \Pi_{2s}(x).
\end{aligned}$$

Use the martingales $\{e^{\alpha_1(x+X(t))}I_{(t<\tau)};\ t \geq 0\}$ and $\{e^{\beta_1(x+X(t))}I_{(t<\tau)};\ t \geq 0\}$, combined with the optional sampling theorem and the memoryless property of the jump random variables, to see that

$$\Pi_{1d}(x)e^{\alpha_1 x_1} + \Pi_{1s}(x)e^{\alpha_1 x_1}\frac{w}{w+\alpha_1} + \Pi_{2d}(x)e^{\alpha_1 x_2} + \Pi_{2s}(x)e^{\alpha_1 x_2}\frac{v}{v-\alpha_1}$$

$$= e^{\alpha_1 x},$$

$$\Pi_{1d}(x)e^{\beta_1 x_1} + \Pi_{1s}(x)e^{\beta_1 x_1}\frac{w}{w+\beta_1} + \Pi_{2d}(x)e^{\beta_1 x_2} + \Pi_{2s}(x)e^{\beta_1 x_2}\frac{v}{v-\beta_1}$$

$$= e^{\beta_1 x}.$$

By analytical analogy we have

$$\Pi_{1d}(x)e^{\alpha_2 x_1} + \Pi_{1s}(x)e^{\alpha_2 x_1}\frac{w}{w+\alpha_2} + \Pi_{2d}(x)e^{\alpha_2 x_2} + \Pi_{2s}(x)e^{\alpha_2 x_2}\frac{v}{v-\alpha_2}$$

$$= e^{\alpha_2 x}$$

$$\Pi_{1d}(x)e^{\beta_2 x_1} + \Pi_{1s}(x)e^{\beta_2 x_1}\frac{w}{w+\beta_2} + \Pi_{2d}(x)e^{\beta_2 x_2} + \Pi_{2s}(x)e^{\beta_2 x_2}\frac{v}{v-\beta_2}$$

$$= e^{\beta_2 x}.$$

- Use Cramer's rule to obtain $\Pi_{1d}(x),\ \Pi_{1s}(x),\ \Pi_{2d}(x),\ \Pi_{2s}(x)$
- Each is a linear combination of $e^{\alpha_1 x},\ e^{\alpha_2 x},\ e^{\beta_1 x}, e^{\beta_2 x}$

Solutions of integro-differential equations

$$
\begin{array}{llll}
\Pi_{1d}(x): & \mathcal{L}\phi(x) = 0, & x_1 \le x \le x_2, & \phi(x_1) = 1,\ \phi(x_2) = 0 \\
\Pi_{2d}(x): & \mathcal{L}\phi(x) = 0, & x_1 \le x \le x_2, & \phi(x_1) = 0,\ \phi(x_2) = 1 \\
\Pi_{1s}(x): & \mathcal{L}\phi(x) + \omega e^{-w(x-x_1)} = 0, & & \phi(x_1) = \phi(x_2) = 0 \\
\Pi_{2s}(x): & \mathcal{L}\phi(x) + \nu e^{-v(x_2-x)} = 0, & & \phi(x_1) = \phi(x_2) = 0
\end{array}
$$

with

$$
\begin{aligned}
\mathcal{L}\phi(x) &= D\phi''(x) + \mu\phi'(x) - (\lambda + \nu + \omega)\phi(x) \\
&\quad + \nu v \int_0^{x_2 - x} \phi(x+y) e^{-vy} dy + \omega w \int_0^{x - x_1} \phi(x-y) e^{-wy} dy
\end{aligned}
$$

Substitute

$$\phi(x) = A_1 e^{\alpha_1 x} + A_2 e^{\alpha_2 x} + B_1 e^{\beta_1 x} + B_2 e^{\beta_2 x}.$$

and get each time a system of four linear equations:

$$\begin{aligned}
A_1 e^{\alpha_1 x_2} \frac{v}{v-\alpha_1} + A_2 e^{\alpha_2 x_2} \frac{v}{v-\alpha_2} + B_1 e^{\beta_1 x_2} \frac{v}{v-\beta_1} + B_2 e^{\beta_2 x_2} \frac{v}{v-\beta_2} &= 0 \\
A_1 e^{\alpha_1 x_1} \frac{w}{w+\alpha_1} + A_2 e^{\alpha_2 x_1} \frac{w}{w+\alpha_2} + B_1 e^{\beta_1 x_1} \frac{w}{w+\beta_1} + B_2 e^{\beta_2 x_1} \frac{w}{w+\beta_2} &= 0 \\
A_1 e^{\alpha_1 x_1} + A_2 e^{\alpha_2 x_1} + B_1 e^{\beta_1 x_1} + B_2 e^{\beta_2 x_1} &= 1 \\
A_1 e^{\alpha_1 x_2} + A_2 e^{\alpha_2 x_2} + B_1 e^{\beta_1 x_2} + B_2 e^{\beta_2 x_2} &= 0
\end{aligned}$$

for $\phi(x) = \Pi_{1d}(x)$.

Substitute

$$\phi(x) = A_1 e^{\alpha_1 x} + A_2 e^{\alpha_2 x} + B_1 e^{\beta_1 x} + B_2 e^{\beta_2 x}.$$

and get each time a system of four linear equations:

$$\begin{aligned}
A_1 e^{\alpha_1 x_2} \frac{v}{v-\alpha_1} + A_2 e^{\alpha_2 x_2} \frac{v}{v-\alpha_2} + B_1 e^{\beta_1 x_2} \frac{v}{v-\beta_1} + B_2 e^{\beta_2 x_2} \frac{v}{v-\beta_2} &= 0 \\
A_1 e^{\alpha_1 x_1} \frac{w}{w+\alpha_1} + A_2 e^{\alpha_2 x_1} \frac{w}{w+\alpha_2} + B_1 e^{\beta_1 x_1} \frac{w}{w+\beta_1} + B_2 e^{\beta_2 x_1} \frac{w}{w+\beta_2} &= 0 \\
A_1 e^{\alpha_1 x_1} + A_2 e^{\alpha_2 x_1} + B_1 e^{\beta_1 x_1} + B_2 e^{\beta_2 x_1} &= 0 \\
A_1 e^{\alpha_1 x_2} + A_2 e^{\alpha_2 x_2} + B_1 e^{\beta_1 x_2} + B_2 e^{\beta_2 x_2} &= 1
\end{aligned}$$

for $\phi(x) = \Pi_{2d}(x)$.

Substitute

$$\phi(x) = A_1 e^{\alpha_1 x} + A_2 e^{\alpha_2 x} + B_1 e^{\beta_1 x} + B_2 e^{\beta_2 x}.$$

and get each time a system of four linear equations:

$$\begin{aligned}
A_1 e^{\alpha_1 x_2} \frac{v}{v-\alpha_1} + A_2 e^{\alpha_2 x_2} \frac{v}{v-\alpha_2} + B_1 e^{\beta_1 x_2} \frac{v}{v-\beta_1} + B_2 e^{\beta_2 x_2} \frac{v}{v-\beta_2} &= 0 \\
A_1 e^{\alpha_1 x_1} \frac{w}{w+\alpha_1} + A_2 e^{\alpha_2 x_1} \frac{w}{w+\alpha_2} + B_1 e^{\beta_1 x_1} \frac{w}{w+\beta_1} + B_2 e^{\beta_2 x_1} \frac{w}{w+\beta_2} &= 1 \\
A_1 e^{\alpha_1 x_1} + A_2 e^{\alpha_2 x_1} + B_1 e^{\beta_1 x_1} + B_2 e^{\beta_2 x_1} &= 0 \\
A_1 e^{\alpha_1 x_2} + A_2 e^{\alpha_2 x_2} + B_1 e^{\beta_1 x_2} + B_2 e^{\beta_2 x_2} &= 0
\end{aligned}$$

for $\phi(x) = \Pi_{1s}(x)$.

Substitute

$$\phi(x) = A_1 e^{\alpha_1 x} + A_2 e^{\alpha_2 x} + B_1 e^{\beta_1 x} + B_2 e^{\beta_2 x}.$$

and get each time a system of four linear equations:

$$\begin{aligned}
A_1 e^{\alpha_1 x_2} \frac{v}{v-\alpha_1} + A_2 e^{\alpha_2 x_2} \frac{v}{v-\alpha_2} + B_1 e^{\beta_1 x_2} \frac{v}{v-\beta_1} + B_2 e^{\beta_2 x_2} \frac{v}{v-\beta_2} &= 1 \\
A_1 e^{\alpha_1 x_1} \frac{w}{w+\alpha_1} + A_2 e^{\alpha_2 x_1} \frac{w}{w+\alpha_2} + B_1 e^{\beta_1 x_1} \frac{w}{w+\beta_1} + B_2 e^{\beta_2 x_1} \frac{w}{w+\beta_2} &= 0 \\
A_1 e^{\alpha_1 x_1} + A_2 e^{\alpha_2 x_1} + B_1 e^{\beta_1 x_1} + B_2 e^{\beta_2 x_1} &= 0 \\
A_1 e^{\alpha_1 x_2} + A_2 e^{\alpha_2 x_2} + B_1 e^{\beta_1 x_2} + B_2 e^{\beta_2 x_2} &= 0
\end{aligned}$$

for $\phi(x) = \Pi_{2s}(x)$.

Double barrier knock-in option

- $0 < L < S(0) < U$
- barrier levels $L, \ U$; initial stock price $S(0)$
- The option comes into existence if one of the two barriers is reached before time τ
- Payoff $I_{(S(0)e^{m(\tau)} \leq L, \text{ or } S(0)e^{M(\tau)} \geq U)} b(S(\tau))$

- Expected payoff of the double-barrier knock-in option is

$$\Pi_{1d}(\ln S(0))\mathcal{E}_b(L) + \Pi_{1s}(\ln S(0)) \int_0^\infty \mathcal{E}_b(Le^{-x}) we^{-wx} \mathrm{d}x$$
$$+\Pi_{2d}(\ln S(0))\mathcal{E}_b(U) + \Pi_{2s}(\ln S(0)) \int_0^\infty \mathcal{E}_b(Ue^{x}) ve^{-vx} \mathrm{d}x,$$

with $x_1 = \ln \frac{L}{S(0)}$ and $x_2 = \ln \frac{U}{S(0)}$.